

Noether theorems in a general setting

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Abstract

The first and second Noether theorems are formulated in a general case of reducible degenerate Grassmann-graded Lagrangian theory of even and odd variables on graded bundles. Such Lagrangian theory is characterized by a hierarchy of non-trivial higher-stage Noether identities and the corresponding higher-stage gauge symmetries which are described in the homology terms. In these terms, the second Noether theorems associate to the Koszul – Tate chain complex of higher-stage Noether identities the gauge cochain sequence whose ascent operator provides higher-order gauge symmetries of Lagrangian theory. If gauge symmetries are algebraically closed, this operator is extended to the nilpotent BRST operator which brings the gauge cochain sequence into the BRST complex. In this framework, the first Noether theorem is formulated as a straightforward corollary of the first variational formula. It associates to any variational Lagrangian symmetry the conserved current whose total differential vanishes on-shell. We prove in a general setting that a conserved current of a gauge symmetry is reduced to a total differential on-shell. The physically relevant examples of gauge theory on principal bundles, gauge gravitational theory on natural bundles, topological Chern – Simons field theory and topological BF theory are present. The last one exemplifies a reducible Lagrangian system.

Contents

1	Introduction	2
2	Graded bundles	4
2.1	Grassmann-graded algebraic calculus	5
2.2	Grassmann-graded differential calculus	6
2.3	Graded manifolds	9
2.4	Graded bundles over smooth manifolds	12
2.5	Graded jet manifolds	14
3	Graded Lagrangian formalism	18
4	First Noether theorem	21
4.1	Infinitesimal graded transformations of Lagrangian systems	21
4.2	Lagrangian symmetries and conservation laws	24
4.3	Gauge symmetries	27
4.4	Gauge conservation laws	28

5	Second Noether theorems	30
5.1	Noether and higher-stage Noether identities	30
5.2	Inverse second Noether theorem	38
5.3	Direct second Noether theorem	43
5.4	BRST operator	44
5.5	Lagrangian BRST theory	46
5.6	Appendix. Noether identities of differential operators	49
6	Classical field models	57
6.1	Gauge theory on principal bundles	57
6.2	Gauge gravitation theory on natural bundles	63
6.3	Chern – Simons topological theory	68
6.4	Topological BF theory	72

1 Introduction

The Noether theorems are well known to treat symmetries of Lagrangian systems [47]. The first Noether theorem associates to a Lagrangian symmetry the conserved current whose total differential vanishes on-shell. The second ones provide the correspondence between Noether identities and gauge symmetries of a Lagrangian system.

We aim to formulate Noether theorems in a general case of reducible degenerate Lagrangian systems characterized by a hierarchy of non-trivial higher-stage Noether identities (Section 5.1). To describe this hierarchy, one need to involve Grassmann-graded variables. In a general setting, we therefore consider Grassmann-graded Lagrangian systems of even and odd variables on a smooth manifold X (Section 3).

Lagrangian theory of even (commutative) variables on an n -dimensional smooth manifold X conventionally is formulated in terms of smooth fibre bundles over X and jet manifolds of their sections [11, 30, 52, 65, 71] in the framework of general technique of non-linear differential operators and equations [17, 30, 48]. This formulation is based on the categorial equivalence of projective $C^\infty(X)$ -modules of finite ranks and vector bundles over X in accordance with the classical Serre – Swan theorem, generalized to non-compact manifolds [36, 53, 55].

At the same time, different geometric models of odd variables either on graded manifolds or supermanifolds are discussed [18, 19, 24, 50, 51, 66]. Both graded manifolds and supermanifolds are phrased in terms of sheaves of graded commutative algebras [4, 36, 62]. However, graded manifolds are characterized by sheaves on smooth manifolds, while supermanifolds are constructed by gluing of sheaves on supervector spaces. Since non-trivial higher-stage Noether identities of a Lagrangian system on a smooth manifold X form graded $C^\infty(X)$ -modules, we follow the above mentioned Serre – Swan theorem extended to graded manifolds (Theorem 2.2) [66, 67]. It states that, if a graded commutative $C^\infty(X)$ -ring is generated by a projective $C^\infty(X)$ -module of finite rank, it is isomorphic to a ring of graded functions on a graded manifold whose body is X . Accordingly, we describe odd variables in terms of graded manifolds too [36, 65, 66].

Let us recall that a graded manifold is a locally-ringed space, characterized by a smooth body manifold Z and some structure sheaf \mathfrak{A} of Grassmann algebras on Z [4, 36, 62]. Its sections

form a graded commutative $C^\infty(Z)$ -ring \mathcal{A} of graded functions on a graded manifold (Z, \mathfrak{A}) . It is called the structure ring of (Z, \mathfrak{A}) . The differential calculus on a graded manifold is defined as the Chevalley – Eilenberg differential calculus over its structure ring (Section 2.3). By virtue of the well-known Batchelor theorem (Theorem 2.1), there exists a vector bundle $E \rightarrow Z$ with a typical fibre V such that the structure sheaf \mathfrak{A} of (Z, \mathfrak{A}) is isomorphic to a sheaf \mathfrak{A}_E of germs of sections of the exterior bundle $\wedge E^*$ of the dual E^* of E whose typical fibre is the Grassmann algebra $\wedge V^*$ [4, 10]. This Batchelor’s isomorphism is not canonical. In applications, it however is fixed from the beginning. Therefore, we restrict our consideration to graded manifolds (Z, \mathfrak{A}_E) , called the simple graded manifolds, modelled over vector bundles $E \rightarrow Z$.

Let us note that a smooth manifold Z itself can be treated as a trivial simple graded manifold (Z, C_Z^∞) modelled over a trivial bundle $Z \times \mathbb{R} \rightarrow Z$ whose structure ring of graded functions is reduced to a ring $C^\infty(Z)$ of smooth real functions on Z (Example 2.3). Accordingly, a fibre bundle $Y \rightarrow X$ in a Lagrangian theory of even variables can be regarded as a graded bundle of trivial graded manifolds $(Y, C_Y^\infty) \rightarrow (X, C_X^\infty)$ (Example 2.4). It follows that, in a general setting, one can define a configuration space of Grassmann-graded Lagrangian theory of even and odd variables as being a graded bundle

$$(Y, \mathfrak{A}_F) \rightarrow (X, C_X^\infty) \quad (1.1)$$

over a trivial graded manifold (X, C_X^∞) (Section 2.4) where (Y, \mathfrak{A}_F) is a simple graded manifold modelled over a vector bundle $F \rightarrow Y$ whose body is a smooth bundle $Y \rightarrow X$ [36, 66, 68]. If $Y \rightarrow X$ is a vector bundle, this is a particular case of graded vector bundles in [28, 51] whose base is a trivial graded manifold.

Lagrangian theory on a fibre bundles $Y \rightarrow X$ can be adequately formulated in algebraic terms of a variational bicomplex of exterior forms on the infinite order jet manifold $J^\infty Y$ of sections of $Y \rightarrow X$, without appealing to the calculus of variations [2, 11, 31, 36, 52, 65, 71]. This technique is extended to Lagrangian theory on graded bundles [3, 8, 9, 34, 36, 66]. It is comprehensively phrased in terms of the Grassmann-graded variational bicomplex (3.2) of graded exterior forms on a graded infinite order jet manifold $(J^\infty Y, \mathcal{A}_{J^\infty F})$ (Section 3). Lagrangians and the Euler – Lagrange operator are defined as the elements (3.5) and the coboundary operator (3.6) of this bicomplex, respectively. The cohomology of the variational bicomplex provides the global variational decomposition (3.11) for Lagrangians and Euler – Lagrange operators (Theorem 3.5).

In these terms, the first Noether theorem is formulated as a straightforward corollary of the variational decomposition (3.11) (Section 4). It associates to any variational symmetry of a Lagrangian L the conserved current (4.17) whose total differential vanishes on-shell (Theorem 4.7). It is important that, in the case of a gauge symmetry, the corresponding conserved current is reduced to a superpotential, i.e., this is a total differential on-shell (Theorem 4.9). This fact was proved in different particular variants [21, 36, 38, 45]. We do this in a very general setting [61].

Given a gauge symmetry of a graded Lagrangian, the direct second Noether theorem (Theorem 5.10) states that the Euler – Lagrange operator obeys the corresponding Noether identities. A problem is that any Euler – Lagrange operator satisfies Noether identities, which therefore must be separated into the trivial and non-trivial ones. These Noether identities can obey first-stage Noether identities, which in turn are subject to the second-stage ones, and so on. Thus, there

is a hierarchy of Noether and higher-stage Noether identities which also must be separated into the trivial and non-trivial ones. In accordance with general analysis of Noether identities of differential operators [36, 57], if certain homology regularity conditions hold (Condition 5.5), one can associate to a Grassmann-graded Lagrangian system the exact Koszul – Tate complex (5.31) possessing the boundary operator whose nilpotentness is equivalent to all complete non-trivial Noether and higher-stage Noether identities (5.15) and (5.32) [8, 9, 36, 66].

It should be noted that the notion of higher-stage Noether identities has come from that of reducible constraints. The Koszul – Tate complex of Noether identities has been invented similarly to that of constraints under the condition that Noether identities are locally separated into independent and dependent ones [3, 23]. This condition is relevant for constraints, defined by a finite set of functions which the inverse mapping theorem is applied to. However, Noether identities unlike constraints are differential equations. They are given by an infinite set of functions on a Fréchet manifold of infinite order jets where the inverse mapping theorem fails to be valid. Therefore, the regularity condition for the Koszul – Tate complex of constraints is replaced with the above mentioned homology regularity condition.

The inverse second Noether theorem formulated in homology terms (Theorem 5.9) associates to this Koszul – Tate complex the cochain sequence (5.42) with the ascent operator (5.43), called the gauge operator, whose components are complete non-trivial gauge and higher-stage gauge symmetries of Lagrangian field theory [9, 36, 66].

The gauge operator unlike the Koszul – Tate one is not nilpotent, unless gauge symmetries are abelian. This is the cause why an intrinsic definition of non-trivial gauge and higher-stage gauge symmetries meets difficulties. Another problem is that gauge symmetries need not form an algebra [26, 35, 37]. Therefore, we replace the notion of the algebra of gauge symmetries with some conditions on the gauge operator. Gauge symmetries are said to be algebraically closed if the gauge operator admits a nilpotent extension, called the BRST (Becchi – Rouet – Stora – Tyutin) operator (Section 5.4). If the BRST operator exists, the above mentioned cochain sequence is brought into the BRST complex. The Koszul – Tate and BRST complexes provide a BRST extension of original Lagrangian theory by Grassmann-graded ghosts and Noether antifields [35, 36, 66].

Classical field theory is formulated adequately as a Lagrangian theory on graded bundles [36, 60, 66]. Section 6 contains some examples of relevant field models: gauge theory on principal bundles, gravitation theory on natural bundles, Chern – Simons topological theory, topological BF theory. The last one exemplifies a reducible Lagrangian system.

2 Graded bundles

Throughout this work, by the Grassmann gradation is meant the \mathbb{Z}_2 -one, and a Grassmann graded structure simply is called the graded one if there is no danger of confusion. Hereafter, the symbol $[\cdot]$ stands for the Grassmann parity.

Smooth manifolds throughout are assumed to be Hausdorff, second-countable and, consequently, paracompact and locally compact, countable at infinity. Given a smooth manifold X , its tangent and cotangent bundles TX and T^*X are endowed with bundle coordinates $(x^\lambda, \dot{x}^\lambda)$ and $(x^\lambda, \dot{x}_\lambda)$ with respect to holonomic frames $\{\partial_\lambda\}$ and $\{dx^\lambda\}$, respectively.

Given a manifold X and its coordinate chart $(U; x^\lambda)$, a multi-index Λ of the length $|\Lambda| = k$ throughout denotes a collection of indices $(\lambda_1 \dots \lambda_k)$ modulo permutations. By $\lambda + \Lambda$ is meant a multi-index $(\lambda \lambda_1 \dots \lambda_k)$. Summation over a multi-index Λ means separate summation over each its index λ_i . We use the compact notation $\partial_\Lambda = \partial_{\lambda_k} \circ \dots \circ \partial_{\lambda_1}$ and $\Lambda = (\lambda_1 \dots \lambda_k)$.

2.1 Grassmann-graded algebraic calculus

Let us summarize the relevant basics of the Grassmann-graded algebraic calculus [4, 36, 62].

Let \mathcal{K} be a commutative ring. A \mathcal{K} -module Q is called graded if it is endowed with a grading automorphism γ , $\gamma^2 = \text{Id}$. A graded module falls into a direct sum of modules $Q = Q_0 \oplus Q_1$ such that $\gamma(q) = (-1)^{[q]}q$, $q \in Q_{[q]}$. One calls Q_0 and Q_1 the even and odd parts of Q , respectively.

In particular, by a real graded vector space $B = B_0 \oplus B_1$ is meant a graded \mathbb{R} -module. It is said to be (n, m) -dimensional if $B_0 = \mathbb{R}^n$ and $B_1 = \mathbb{R}^m$.

A \mathcal{K} -algebra \mathcal{A} is called graded if it is a graded \mathcal{K} -module such that $[aa'] = ([a] + [a']) \bmod 2$, where a and a' are graded-homogeneous elements of \mathcal{A} . Its even part \mathcal{A}_0 is a subalgebra of \mathcal{A} , and the odd one \mathcal{A}_1 is an \mathcal{A}_0 -module. If \mathcal{A} is a graded ring with the unit $\mathbf{1}$, then $[\mathbf{1}] = 0$.

A graded algebra \mathcal{A} is called graded commutative if $aa' = (-1)^{[a][a']}a'a$.

Example 2.1. Let V be a real vector space, and let $\Lambda = \wedge V$ be its exterior algebra endowed with the Grassmann gradation

$$\Lambda = \Lambda_0 \oplus \Lambda_1, \quad \Lambda_0 = \mathbb{R} \bigoplus_{k=1}^{2k} \wedge^k V, \quad \Lambda_1 = \bigoplus_{k=1}^{2k-1} \wedge^k V. \quad (2.1)$$

It is a real graded commutative ring, called the Grassmann algebra. A Grassmann algebra, seen as an additive group, admits the decomposition

$$\Lambda = \mathbb{R} \oplus R = \mathbb{R} \oplus R_0 \oplus R_1 = \mathbb{R} \oplus (\Lambda_1)^2 \oplus \Lambda_1, \quad (2.2)$$

where R is the ideal of nilpotents of Λ . The corresponding epimorphism $\sigma : \Lambda \rightarrow \mathbb{R}$ is called the body map. Note that there is a different definition of a Grassmann algebra [44]. Hereafter, we restrict our consideration to Grassmann algebras of finite rank when $V = \mathbb{R}^N$. Given a basis $\{c^i\}$ for V , elements of the Grassmann algebra Λ (2.1) take a form

$$a = \sum_{k=0,1,\dots} \sum_{(i_1 \dots i_k)} a_{i_1 \dots i_k} c^{i_1} \dots c^{i_k}, \quad (2.3)$$

where the second sum runs through all the tuples $(i_1 \dots i_k)$ such that no two of them are permutations of each other.

Given a graded algebra \mathcal{A} , a left graded \mathcal{A} -module Q is defined as a left \mathcal{A} -module where $[aq] = ([a] + [q]) \bmod 2$. Similarly, right graded \mathcal{A} -modules are treated.

Example 2.2. A graded algebra \mathfrak{g} is called a Lie superalgebra if its product $[\cdot, \cdot]$, called the Lie superbracket, obeys the relations

$$\begin{aligned} [\varepsilon, \varepsilon'] &= -(-1)^{[\varepsilon][\varepsilon']}[\varepsilon', \varepsilon], \\ (-1)^{[\varepsilon][\varepsilon'']}[\varepsilon, [\varepsilon', \varepsilon'']] &+ (-1)^{[\varepsilon'][\varepsilon]}[\varepsilon', [\varepsilon'', \varepsilon]] + (-1)^{[\varepsilon''][\varepsilon']}[\varepsilon'', [\varepsilon, \varepsilon']] = 0. \end{aligned}$$

Being decomposed in even and odd parts $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, a Lie superalgebra \mathfrak{g} obeys the relations

$$[\mathfrak{g}_0, \mathfrak{g}_0] \subset \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_1] \subset \mathfrak{g}_1, \quad [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0.$$

In particular, an even part \mathfrak{g}_0 of a Lie superalgebra \mathfrak{g} is a Lie algebra. A graded vector space P is a \mathfrak{g} -module if it is provided with an \mathbb{R} -bilinear map

$$\begin{aligned} \mathfrak{g} \times P \ni (\varepsilon, p) &\rightarrow \varepsilon p \in P, & [\varepsilon p] &= ([\varepsilon] + [p]) \bmod 2, \\ [\varepsilon, \varepsilon'] p &= (\varepsilon \circ \varepsilon' - (-1)^{[\varepsilon][\varepsilon']} \varepsilon' \circ \varepsilon) p. \end{aligned}$$

If \mathcal{A} is graded commutative, a graded \mathcal{A} -module Q is provided with a graded \mathcal{A} -bimodule structure by letting $qa = (-1)^{[a][q]}aq$, $a \in \mathcal{A}$, $q \in Q$.

Given a graded commutative ring \mathcal{A} , the following are standard constructions of new graded modules from the old ones.

- The direct sum of graded modules and a graded factor module are defined just as those of modules over a commutative ring.
- The tensor product $P \otimes Q$ of graded \mathcal{A} -modules P and Q is their tensor product as \mathcal{A} -modules such that

$$\begin{aligned} [p \otimes q] &= [p] + [q], & p &\in P, & q &\in Q, \\ ap \otimes q &= (-1)^{[p][a]}pa \otimes q = (-1)^{[p][a]}p \otimes aq, & a &\in \mathcal{A}. \end{aligned}$$

In particular, the tensor algebra $\otimes P$ of a graded \mathcal{A} -module P is defined just as that of a module over a commutative ring. Its quotient $\wedge P$ with respect to the ideal generated by elements

$$p \otimes p' + (-1)^{[p][p']} p' \otimes p, \quad p, p' \in P,$$

is a bigraded exterior algebra of a graded module P provided with the graded exterior product

$$p \wedge p' = -(-1)^{[p][p']} p' \wedge p.$$

- A morphism $\Phi : P \rightarrow Q$ of graded \mathcal{A} -modules seen as additive groups is said to be an even (resp. odd) graded morphism if Φ preserves (resp. changes) the Grassmann parity of all graded-homogeneous elements of P , and if the relations

$$\Phi(ap) = (-1)^{[\Phi][a]}a\Phi(p), \quad p \in P, \quad a \in \mathcal{A},$$

hold. A morphism $\Phi : P \rightarrow Q$ of graded \mathcal{A} -modules as additive groups is called a graded \mathcal{A} -module morphism if it is represented by a sum of even and odd graded morphisms. A set $\text{Hom}_{\mathcal{A}}(P, Q)$ of graded morphisms of a graded \mathcal{A} -module P to a graded \mathcal{A} -module Q is naturally a graded \mathcal{A} -module. A graded \mathcal{A} -module $P^* = \text{Hom}_{\mathcal{A}}(P, \mathcal{A})$ is called the dual of a graded \mathcal{A} -module P .

2.2 Grassmann-graded differential calculus

Linear differential operators and the differential calculus over a graded commutative ring are defined similarly to those in commutative geometry [36, 62, 64].

Let \mathcal{K} be a commutative ring and \mathcal{A} a graded commutative \mathcal{K} -ring. Let P and Q be graded \mathcal{A} -modules. A \mathcal{K} -module $\text{Hom}_{\mathcal{K}}(P, Q)$ of graded \mathcal{K} -module homomorphisms $\Phi : P \rightarrow Q$ can be endowed with the two graded \mathcal{A} -module structures

$$(a\Phi)(p) = a\Phi(p), \quad (\Phi \bullet a)(p) = \Phi(ap), \quad a \in \mathcal{A}, \quad p \in P.$$

Let us put

$$\delta_a \Phi = a\Phi - (-1)^{[a][\Phi]} \Phi \bullet a, \quad a \in \mathcal{A}. \quad (2.4)$$

An element $\Delta \in \text{Hom}_{\mathcal{K}}(P, Q)$ is said to be a Q -valued graded differential operator of order s on P if $\delta_{a_0} \circ \dots \circ \delta_{a_s} \Delta = 0$ for any tuple of $s+1$ elements a_0, \dots, a_s of \mathcal{A} .

In particular, zero order graded differential operators coincide with graded \mathcal{A} -module morphisms $P \rightarrow Q$. A first order graded differential operator Δ satisfies a relation

$$\begin{aligned} \delta_a \circ \delta_b \Delta(p) &= ab\Delta(p) - (-1)^{([b]+[\Delta])[a]} b\Delta(ap) - (-1)^{[b][\Delta]} a\Delta(bp) + \\ &(-1)^{[b][\Delta]+([\Delta]+[b])[a]} = 0, \quad a, b \in \mathcal{A}, \quad p \in P. \end{aligned}$$

For instance, let $P = \mathcal{A}$. Any zero order Q -valued graded differential operator Δ on \mathcal{A} is defined by its value $\Delta(\mathbf{1})$. A first order Q -valued graded differential operator Δ on \mathcal{A} fulfils a condition

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]} a\Delta(b) - (-1)^{([b]+[a])[\Delta]} ab\Delta(\mathbf{1}), \quad a, b \in \mathcal{A}.$$

It is called the Q -valued graded derivation of \mathcal{A} if $\Delta(\mathbf{1}) = 0$, i.e., the graded Leibniz rule

$$\Delta(ab) = \Delta(a)b + (-1)^{[a][\Delta]} a\Delta(b), \quad a, b \in \mathcal{A}, \quad (2.5)$$

holds. One then observes that any first order graded differential operator on \mathcal{A} falls into a sum

$$\Delta(a) = \Delta(\mathbf{1})a + [\Delta(a) - \Delta(\mathbf{1})a]$$

of a zero order graded differential operator $\Delta(\mathbf{1})a$ and a graded derivation $\Delta(a) - \Delta(\mathbf{1})a$. If ∂ is a graded derivation of \mathcal{A} , then $a\partial$ is so for any $a \in \mathcal{A}$. Hence, graded derivations of \mathcal{A} constitute a graded \mathcal{A} -module $\mathfrak{d}(\mathcal{A}, Q)$, called the graded derivation module. If $Q = \mathcal{A}$, a graded derivation module $\mathfrak{d}\mathcal{A}$ also is a Lie superalgebra over a commutative ring \mathcal{K} with respect to a superbracket

$$[u, u'] = u \circ u' - (-1)^{[u][u']} u' \circ u, \quad u, u' \in \mathcal{A}. \quad (2.6)$$

Since $\mathfrak{d}\mathcal{A}$ is a Lie \mathcal{K} -superalgebra, let us consider the Chevalley – Eilenberg complex $C^*[\mathfrak{d}\mathcal{A}; \mathcal{A}]$ where a graded commutative ring \mathcal{A} is regarded as a $\mathfrak{d}\mathcal{A}$ -module [25, 36, 64]. It is the complex

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{A} \xrightarrow{d} C^1[\mathfrak{d}\mathcal{A}; \mathcal{A}] \xrightarrow{d} \dots C^k[\mathfrak{d}\mathcal{A}; \mathcal{A}] \xrightarrow{d} \dots \quad (2.7)$$

where

$$C^k[\mathfrak{d}\mathcal{A}; \mathcal{A}] = \text{Hom}_{\mathcal{K}}(\bigwedge^k \mathfrak{d}\mathcal{A}, \mathcal{A})$$

are $\mathfrak{d}\mathcal{A}$ -modules of \mathcal{K} -linear graded morphisms of graded exterior products $\bigwedge^k \mathfrak{d}\mathcal{A}$ of a graded \mathcal{K} -module $\mathfrak{d}\mathcal{A}$ to \mathcal{A} . Let us bring homogeneous elements of $\bigwedge^k \mathfrak{d}\mathcal{A}$ into the form

$$\varepsilon_1 \wedge \dots \wedge \varepsilon_r \wedge \varepsilon_{r+1} \wedge \dots \wedge \varepsilon_k, \quad \varepsilon_i \in \mathfrak{d}\mathcal{A}_0, \quad \varepsilon_j \in \mathfrak{d}\mathcal{A}_1.$$

Then the Chevalley – Eilenberg coboundary operator d of the complex (2.7) is given by the expression

$$\begin{aligned}
dc(\varepsilon_1 \wedge \cdots \wedge \varepsilon_r \wedge \epsilon_1 \wedge \cdots \wedge \epsilon_s) = & \quad (2.8) \\
& \sum_{i=1}^r (-1)^{i-1} \varepsilon_i c(\varepsilon_1 \wedge \cdots \widehat{\varepsilon_i} \cdots \wedge \varepsilon_r \wedge \epsilon_1 \wedge \cdots \wedge \epsilon_s) + \\
& \sum_{j=1}^s (-1)^r \varepsilon_i c(\varepsilon_1 \wedge \cdots \wedge \varepsilon_r \wedge \epsilon_1 \wedge \cdots \widehat{\epsilon_j} \cdots \wedge \epsilon_s) + \\
& \sum_{1 \leq i < j \leq r} (-1)^{i+j} c([\varepsilon_i, \varepsilon_j] \wedge \varepsilon_1 \wedge \cdots \widehat{\varepsilon_i} \cdots \widehat{\varepsilon_j} \cdots \wedge \varepsilon_r \wedge \epsilon_1 \wedge \cdots \wedge \epsilon_s) + \\
& \sum_{1 \leq i < j \leq s} c([\epsilon_i, \epsilon_j] \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_r \wedge \epsilon_1 \wedge \cdots \widehat{\epsilon_i} \cdots \widehat{\epsilon_j} \cdots \wedge \epsilon_s) + \\
& \sum_{1 \leq i < r, 1 \leq j \leq s} (-1)^{i+r+1} c([\varepsilon_i, \epsilon_j] \wedge \varepsilon_1 \wedge \cdots \widehat{\varepsilon_i} \cdots \wedge \varepsilon_r \wedge \epsilon_1 \wedge \cdots \widehat{\epsilon_j} \cdots \wedge \epsilon_s),
\end{aligned}$$

where the caret $\widehat{}$ denotes omission.

It is easily justified that the complex (2.7) contains a subcomplex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ of \mathcal{A} -linear graded morphisms. The \mathbb{N} -graded module $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is provided with the structure of a bigraded \mathcal{A} -algebra with respect to the graded exterior product

$$\begin{aligned}
\phi \wedge \phi'(u_1, \dots, u_{r+s}) &= \sum_{i_1 < \cdots < i_r; j_1 < \cdots < j_s} \text{Sgn}_{1 \dots r+s}^{i_1 \dots i_r j_1 \dots j_s} \phi(u_{i_1}, \dots, u_{i_r}) \phi'(u_{j_1}, \dots, u_{j_s}), \quad (2.9) \\
\phi &\in \mathcal{O}^r[\mathfrak{d}\mathcal{A}], \quad \phi' \in \mathcal{O}^s[\mathfrak{d}\mathcal{A}], \quad u_k \in \mathfrak{d}\mathcal{A},
\end{aligned}$$

where u_1, \dots, u_{r+s} are graded-homogeneous elements of $\mathfrak{d}\mathcal{A}$ and

$$u_1 \wedge \cdots \wedge u_{r+s} = \text{Sgn}_{1 \dots r+s}^{i_1 \dots i_r j_1 \dots j_s} u_{i_1} \wedge \cdots \wedge u_{i_r} \wedge u_{j_1} \wedge \cdots \wedge u_{j_s}.$$

The graded Chevalley – Eilenberg coboundary operator d (2.8) and the graded exterior product \wedge (2.9) bring $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ into a differential bigraded algebra (henceforth DBGA) whose elements obey relations

$$\phi \wedge \phi' = (-1)^{|\phi||\phi'| + |\phi|[\phi']} \phi' \wedge \phi, \quad d(\phi \wedge \phi') = d\phi \wedge \phi' + (-1)^{|\phi|} \phi \wedge d\phi'. \quad (2.10)$$

It is called the graded differential calculus over a graded commutative \mathcal{K} -ring \mathcal{A} . In particular, we have

$$\mathcal{O}^1[\mathfrak{d}\mathcal{A}] = \text{Hom}_{\mathcal{A}}(\mathfrak{d}\mathcal{A}, \mathcal{A}) = \mathfrak{d}\mathcal{A}^*. \quad (2.11)$$

One can extend this duality relation to the graded interior product of $u \in \mathfrak{d}\mathcal{A}$ with any element $\phi \in \mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ by the rules

$$\begin{aligned}
u \rfloor (bda) &= (-1)^{[u][b]} bu(a), \quad a, b \in \mathcal{A}, \\
u \rfloor (\phi \wedge \phi') &= (u \rfloor \phi) \wedge \phi' + (-1)^{|\phi| + [\phi][u]} \phi \wedge (u \rfloor \phi').
\end{aligned} \quad (2.12)$$

As a consequence, any graded derivation $u \in \mathfrak{d}\mathcal{A}$ of \mathcal{A} yields a derivation

$$\begin{aligned}
\mathbf{L}_u \phi &= u \rfloor d\phi + d(u \rfloor \phi), \quad \phi \in \mathcal{O}^*[\mathfrak{d}\mathcal{A}], \quad u \in \mathfrak{d}\mathcal{A}, \\
\mathbf{L}_u (\phi \wedge \phi') &= \mathbf{L}_u (\phi) \wedge \phi' + (-1)^{[u][\phi]} \phi \wedge \mathbf{L}_u (\phi'),
\end{aligned} \quad (2.13)$$

called the graded Lie derivative of the DBGA $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$.

Note that, if \mathcal{A} is a commutative ring, the graded Chevalley – Eilenberg differential calculus comes to the familiar one.

The minimal graded differential calculus $\mathcal{O}^*\mathcal{A} \subset \mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ over a graded commutative ring \mathcal{A} consists of the monomials $a_0 da_1 \wedge \cdots \wedge da_k$, $a_i \in \mathcal{A}$. The corresponding complex

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{A} \xrightarrow{d} \mathcal{O}^1\mathcal{A} \xrightarrow{d} \cdots \mathcal{O}^k\mathcal{A} \xrightarrow{d} \cdots \quad (2.14)$$

is called the bigraded de Rham complex of a graded commutative \mathcal{K} -ring \mathcal{A} .

2.3 Graded manifolds

A graded manifold of dimension (n, m) is defined as a local-ringed space (Z, \mathfrak{A}) where Z is an n -dimensional smooth manifold Z and $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ is a sheaf of Grassmann algebras Λ of rank m (see Example 2.1) such that [4, 36, 62, 64, 65]:

- there is the exact sequence of sheaves

$$0 \rightarrow \mathcal{R} \rightarrow \mathfrak{A} \xrightarrow{\sigma} C_Z^\infty \rightarrow 0, \quad \mathcal{R} = \mathfrak{A}_1 + (\mathfrak{A}_1)^2, \quad (2.15)$$

where σ is a body epimorphism onto a sheaf C_Z^∞ of smooth real functions on Z ;

- $\mathcal{R}/\mathcal{R}^2$ is a locally free sheaf of C_Z^∞ -modules of finite rank (with respect to pointwise operations), and the sheaf \mathfrak{A} is locally isomorphic to the exterior product $\wedge_{C_Z^\infty}(\mathcal{R}/\mathcal{R}^2)$.

The sheaf \mathfrak{A} is called a structure sheaf of a graded manifold (Z, \mathfrak{A}) , and a manifold Z is said to be the body of (Z, \mathfrak{A}) . Sections of the sheaf \mathfrak{A} are called graded functions on a graded manifold (Z, \mathfrak{A}) . They make up a graded commutative $C^\infty(Z)$ -ring $\mathfrak{A}(Z)$ called the structure ring of (Z, \mathfrak{A}) .

By virtue of the well-known Batchelor theorem [4, 10], graded manifolds possess the following structure.

Theorem 2.1. *Let (Z, \mathfrak{A}) be a graded manifold. There exists a vector bundle $E \rightarrow Z$ with an m -dimensional typical fibre V such that the structure sheaf \mathfrak{A} of (Z, \mathfrak{A}) is isomorphic to the structure sheaf \mathfrak{A}_E of germs of sections of the exterior bundle*

$$\wedge E = Z \times \mathbb{R} \oplus_Z E \oplus_Z^2 E \oplus_Z \cdots \wedge^k E, \quad k = \dim E - \dim Z, \quad (2.16)$$

whose typical fibre is a Grassmann algebra $\wedge V^*$.

Combining Theorem 2.1 and the above mentioned classical Serre – Swan theorem leads to the following Serre – Swan theorem for graded manifolds [8, 67].

Theorem 2.2. *Let Z be a smooth manifold. A graded commutative $C^\infty(Z)$ -algebra \mathcal{A} is isomorphic to the structure ring of a graded manifold with a body Z iff it is the exterior algebra of some projective $C^\infty(Z)$ -module of finite rank.*

Proof. By virtue of the Batchelor theorem, any graded manifold is isomorphic to a simple graded manifold (Z, \mathfrak{A}_E) modelled over some vector bundle $E \rightarrow Z$. Its structure ring \mathcal{A}_E (2.17) of graded functions consists of sections of the exterior bundle $\wedge E^*$ (2.16). The classical Serre – Swan theorem states that a $C^\infty(Z)$ -module is isomorphic to the module of sections of a smooth vector bundle over Z iff it is a projective module of finite rank. \square

It should be emphasized that Batchelor's isomorphism in Theorem 2.1 fails to be canonical. We agree to call (Z, \mathfrak{A}_E) in Theorem 2.1 the simple graded manifold modelled over a characteristic vector bundle $E \rightarrow Z$. Accordingly, a structure ring $\mathfrak{A}_E(Z)$ of a simple graded manifold (Z, \mathfrak{A}_E) is a structure module

$$\mathcal{A}_E = \mathfrak{A}_E(Z) = \wedge E^*(Z) \quad (2.17)$$

of sections of the exterior bundle $\wedge E^*$.

Example 2.3. One can treat a local-ringed space $(Z, \mathfrak{A}_0 = C_Z^\infty)$ as a trivial graded manifold. It is a simple graded manifold whose characteristic bundle is $E = Z \times \{0\}$. Its structure module is a ring $C^\infty(Z)$ of smooth real functions on Z .

Given a simple graded manifold (Z, \mathfrak{A}_E) , every trivialization chart $(U; z^A, q^a)$ of a vector bundle $E \rightarrow Z$ yields a splitting domain $(U; z^A, c^a)$ of (Z, \mathfrak{A}_E) where $\{c^a\}$ is the corresponding local fibre basis for $E^* \rightarrow Z$, i.e., c^a are locally constant sections of $E^* \rightarrow Z$ such that $q_b \circ c^a = \delta_b^a$. Graded functions on such a chart are Λ -valued functions

$$f = \sum_{k=0}^m \frac{1}{k!} f_{a_1 \dots a_k}(z) c^{a_1} \dots c^{a_k}, \quad (2.18)$$

where $f_{a_1 \dots a_k}(z)$ are smooth functions on U . One calls $\{z^A, c^a\}$ the local generating basis for a graded manifold (Z, \mathfrak{A}_E) . Transition functions $q'^a = \rho_b^a(z^A) q^b$ of bundle coordinates on $E \rightarrow Z$ induce the corresponding transformation $c'^a = \rho_b^a(z^A) c^b$ of the associated local generating basis for a graded manifold (Z, \mathfrak{A}_E) and the according coordinate transformation law of graded functions (2.18).

Let us consider the graded derivation module $\mathfrak{d}\mathfrak{A}(Z)$ of a real graded commutative ring $\mathfrak{A}(Z)$. It is a real Lie superalgebra relative to the superbracket (2.6). Its elements are called the graded vector fields on a graded manifold (Z, \mathfrak{A}) . A key point is the following.

Lemma 2.3. *Graded vector fields $u \in \mathfrak{d}\mathcal{A}_E$ on a simple graded manifold (Z, \mathfrak{A}_E) are represented by sections of some vector bundle as follows [36, 62, 65].*

Proof. Due to the canonical splitting $VE = E \times E$, the vertical tangent bundle VE of $E \rightarrow Z$ can be provided with the fibre bases $\{\partial/\partial c^a\}$, which are the duals of the bases $\{c^a\}$. Then graded vector fields on a splitting domain $(U; z^A, c^a)$ of (Z, \mathfrak{A}_E) read

$$\begin{aligned} u &= u^A \partial_A + u^a \frac{\partial}{\partial c^a}, \\ u'^A &= u^A, \quad u'^a = \rho_j^a u^j + u^A \partial_A (\rho_j^a) c^j, \\ \frac{\partial}{\partial c^a} \circ \frac{\partial}{\partial c^b} &= -\frac{\partial}{\partial c^b} \circ \frac{\partial}{\partial c^a}, \quad \partial_A \circ \frac{\partial}{\partial c^a} = \frac{\partial}{\partial c^a} \circ \partial_A. \end{aligned} \quad (2.19)$$

where u^A, u^a are local graded functions on U , and they act on graded functions $f \in \mathfrak{A}_E(U)$ (2.18) by the rule

$$u(f_{a \dots b} c^a \dots c^b) = u^A \partial_A (f_{a \dots b}) c^a \dots c^b + u^k f_{a \dots b} \frac{\partial}{\partial c^k} (c^a \dots c^b). \quad (2.20)$$

This rule implies the corresponding coordinate transformation law

$$u'^A = u^A, \quad u'^a = \rho_j^a u^j + u^A \partial_A (\rho_j^a) c^j$$

of graded vector fields. It follows that graded vector fields (2.19) can be represented by sections of a vector bundle \mathcal{V}_E which is locally isomorphic to a vector bundle $\wedge E^* \otimes_Z (E \oplus TZ)$. \square

Given a structure ring \mathcal{A}_E of graded functions on a simple graded manifold (Z, \mathfrak{A}_E) and the real Lie superalgebra $\mathfrak{d}\mathcal{A}_E$ of its graded derivations, let us consider the graded differential calculus

$$\mathcal{S}^*[E; Z] = \mathcal{O}^*[\mathfrak{d}\mathcal{A}_E] \quad (2.21)$$

over \mathcal{A}_E where $\mathcal{S}^0[E; Z] = \mathcal{A}_E$.

Lemma 2.4. *Since the graded derivation module $\mathfrak{d}\mathcal{A}_E$ is isomorphic to the structure module of sections of a vector bundle $\mathcal{V}_E \rightarrow Z$ in Lemma 2.3, elements of $\mathcal{S}^*[E; Z]$ are represented by sections of the exterior bundle $\wedge \overline{\mathcal{V}}_E$ of the \mathcal{A}_E -dual $\overline{\mathcal{V}}_E \rightarrow Z$ of \mathcal{V}_E .*

With respect to the dual fibre bases $\{dz^A\}$ for T^*Z and $\{dc^b\}$ for E^* , sections of $\overline{\mathcal{V}}_E$ take a coordinate form

$$\begin{aligned} \phi &= \phi_A dz^A + \phi_a dc^a, \\ \phi'_a &= \rho^{-1b}_a \phi_b, \quad \phi'_A = \phi_A + \rho^{-1b}_a \partial_A(\rho^a_j) \phi_b c^j. \end{aligned}$$

The duality isomorphism $\mathcal{S}^1[E; Z] = \mathfrak{d}\mathcal{A}_E^*$ (2.11) is given by the graded interior product

$$u \rfloor \phi = u^A \phi_A + (-1)^{[\phi_a]} u^a \phi_a.$$

Elements of $\mathcal{S}^*[E; Z]$ are called graded exterior forms on a graded manifold (Z, \mathfrak{A}_E) . In particular, elements of $\mathcal{S}^*[E; Z]$ are graded functions on (Z, \mathfrak{A}_E) .

Seen as an \mathcal{A}_E -algebra, the DBGA $\mathcal{S}^*[E; Z]$ (2.21) on a splitting domain $(U; z^A, c^a)$ is locally generated by graded one-forms dz^A , dc^i such that

$$dz^A \wedge dc^i = -dc^i \wedge dz^A, \quad dc^i \wedge dc^j = dc^j \wedge dc^i.$$

Accordingly, the graded Chevalley – Eilenberg coboundary operator d (2.8), called the graded exterior differential, reads

$$d\phi = dz^A \wedge \partial_A \phi + dc^a \wedge \frac{\partial}{\partial c^a} \phi,$$

where derivatives ∂_λ , $\partial/\partial c^a$ act on coefficients of graded exterior forms by the formula (2.20), and they are graded commutative with graded forms dz^A and dc^a . The formulas (2.10) – (2.13) hold.

Lemma 2.5. *The DBGA $\mathcal{S}^*[E; Z]$ (2.21) is a minimal differential calculus over \mathcal{A}_E , i.e., it is generated by elements df , $f \in \mathcal{A}_E$ [36].*

Proof. Since $\mathfrak{d}\mathcal{A}_E = \mathcal{V}_E(Z)$, this is a projective $C^\infty(Z)$ - and \mathcal{A}_E -module of finite rank, and so is its \mathcal{A}_E -dual $\mathcal{S}^1[E; Z]$. Hence, $\mathfrak{d}\mathcal{A}_E$ is the \mathcal{A}_E -dual of $\mathcal{S}^1[E; Z]$ and, consequently, $\mathcal{S}^1[E; Z]$ is generated by elements df , $f \in \mathcal{A}_E$. \square

The bigraded de Rham complex (2.14) of the minimal graded differential calculus $\mathcal{S}^*[E; Z]$ reads

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{A}_E \xrightarrow{d} \mathcal{S}^1[E; Z] \xrightarrow{d} \dots \mathcal{S}^k[E; Z] \xrightarrow{d} \dots \quad (2.22)$$

Its cohomology $H^*(\mathcal{A}_E)$ is called the de Rham cohomology of a simple graded manifold (Z, \mathfrak{A}_E) .

In particular, given the differential graded algebra $\mathcal{O}^*(Z)$ of exterior forms on Z , there exists a canonical monomorphism

$$\mathcal{O}^*(Z) \rightarrow \mathcal{S}^*[E; Z] \quad (2.23)$$

and a body epimorphism $\mathcal{S}^*[E; Z] \rightarrow \mathcal{O}^*(Z)$ which are cochain morphisms of the de Rham complex (2.22) and that of $\mathcal{O}^*(Z)$. Then one can show the following [36, 59].

Theorem 2.6. *The de Rham cohomology of a graded manifold (Z, \mathfrak{A}_E) equals the de Rham cohomology of its body Z .*

Proof. Let \mathfrak{A}_E^k denote the sheaf of germs of graded k -forms on (Z, \mathfrak{A}_E) . Its structure module is $\mathcal{S}^k[E; Z]$. These sheaves constitute the complex

$$0 \rightarrow \mathbb{R} \rightarrow \mathfrak{A}_E \xrightarrow{d} \mathfrak{A}_E^1 \xrightarrow{d} \cdots \mathfrak{A}_E^k \xrightarrow{d} \cdots \quad (2.24)$$

Its members \mathfrak{A}_E^k are sheaves of C_Z^∞ -modules on Z and, consequently, are fine and acyclic. Furthermore, the Poincaré lemma for graded exterior forms holds [4]. It follows that the complex (2.24) is a fine resolution of the constant sheaf \mathbb{R} on a manifold Z . Then, by virtue of the abstract de Rham theorem [36, 41], there is an isomorphism

$$H^*(\mathcal{A}_E) = H^*(Z; \mathbb{R}) = H_{\text{DR}}^*(Z) \quad (2.25)$$

of the cohomology $H^*(\mathcal{A}_E)$ to the de Rham cohomology $H_{\text{DR}}^*(Z)$ of a smooth manifold Z . \square

Corollary 2.7. *The cohomology isomorphism (2.25) accompanies the cochain monomorphism (2.23). Hence, any closed graded exterior form is decomposed into a sum $\phi = \sigma + d\xi$ where σ is a closed exterior form on Z .*

2.4 Graded bundles over smooth manifolds

A morphism of graded manifolds $(Z, \mathfrak{A}) \rightarrow (Z', \mathfrak{A}')$ is defined as that of local-ringed spaces

$$\phi : Z \rightarrow Z', \quad \widehat{\Phi} : \mathfrak{A}' \rightarrow \phi_* \mathfrak{A}, \quad (2.26)$$

where ϕ is a manifold morphism and $\widehat{\Phi}$ is a sheaf morphism of \mathfrak{A}' to the direct image $\phi_* \mathfrak{A}$ of \mathfrak{A} onto Z' [33, 72]. The morphism (2.26) of graded manifolds is said to be:

- a monomorphism if ϕ is an injection and $\widehat{\Phi}$ is an epimorphism;
- an epimorphism if ϕ is a surjection and $\widehat{\Phi}$ is a monomorphism.

An epimorphism of graded manifolds $(Z, \mathfrak{A}) \rightarrow (Z', \mathfrak{A}')$ where $Z \rightarrow Z'$ is a fibre bundle is called the graded bundle [28, 70]. In this case, a sheaf monomorphism $\widehat{\Phi}$ induces a monomorphism of canonical presheaves $\overline{\mathfrak{A}'} \rightarrow \overline{\mathfrak{A}}$, which associates to each open subset $U \subset Z$ the ring of sections of \mathfrak{A}' over $\phi(U)$. Accordingly, there is a pull-back monomorphism of the structure rings $\mathfrak{A}'(Z') \rightarrow \mathfrak{A}(Z)$ of graded functions on graded manifolds (Z', \mathfrak{A}') and (Z, \mathfrak{A}) .

In particular, let (Y, \mathfrak{A}) be a graded manifold whose body $Z = Y$ is a fibre bundle $\pi : Y \rightarrow X$. Let us consider the trivial graded manifold $(X, \mathfrak{A}_0 = C_X^\infty)$ (see Example 2.3). Then we have a graded bundle

$$(Y, \mathfrak{A}) \rightarrow (X, C_X^\infty). \quad (2.27)$$

We agree to call the graded bundle (2.27) over a trivial graded manifold (X, C_X^∞) the graded bundle over a smooth manifold [68]. Let us denote it by (X, Y, \mathfrak{A}) . Given a graded bundle (X, Y, \mathfrak{A}) , the local generating basis for a graded manifold (Y, \mathfrak{A}) can be brought into a form (x^λ, y^i, c^a) where (x^λ, y^i) are bundle coordinates of $Y \rightarrow X$.

If $Y \rightarrow X$ is a vector bundle, the graded bundle (2.27) is a particular case of graded fibre bundles in [28, 51] when their base is a trivial graded manifold.

Example 2.4. Let $Y \rightarrow X$ be a fibre bundle. Then a trivial graded manifold (Y, C_Y^∞) together with a real ring monomorphism $C^\infty(X) \rightarrow C^\infty(Y)$ is the graded bundle (X, Y, C_Y^∞) (2.27).

Example 2.5. A graded manifold (X, \mathfrak{A}) itself can be treated as the graded bundle (X, X, \mathfrak{A}) (2.27) associated to the identity smooth bundle $X \rightarrow X$.

Let $E \rightarrow Z$ and $E' \rightarrow Z'$ be vector bundles and $\Phi : E \rightarrow E'$ their bundle morphism over a morphism $\phi : Z \rightarrow Z'$. Then every section s^* of the dual bundle $E'^* \rightarrow Z'$ defines the pull-back section $\Phi^* s^*$ of the dual bundle $E^* \rightarrow Z$ by the law

$$v_z \rfloor \Phi^* s^*(z) = \Phi(v_z) \rfloor s^*(\phi(z)), \quad v_z \in E_z.$$

It follows that a bundle morphism (Φ, ϕ) yields a morphism of simple graded manifolds

$$(Z, \mathfrak{A}_E) \rightarrow (Z', \mathfrak{A}_{E'}). \quad (2.28)$$

This is a pair $(\phi, \widehat{\Phi} = \phi_* \circ \Phi^*)$ of a morphism ϕ of body manifolds and the composition $\phi_* \circ \Phi^*$ of the pull-back $\mathcal{A}_{E'} \ni f \rightarrow \Phi^* f \in \mathcal{A}_E$ of graded functions and the direct image ϕ_* of a sheaf \mathfrak{A}_E onto Z' . Relative to local bases (z^A, c^a) and (z'^A, c'^a) for (Z, \mathfrak{A}_E) and $(Z', \mathfrak{A}_{E'})$, the morphism (2.28) of simple graded manifolds reads $z' = \phi(z)$, $\widehat{\Phi}(c'^a) = \Phi_b^a(z) c^b$.

The graded manifold morphism (2.28) is a monomorphism (resp. epimorphism) if Φ is a bundle injection (resp. surjection).

In particular, the graded manifold morphism (2.28) is a graded bundle if Φ is a fibre bundle. Let $\mathcal{A}_{E'} \rightarrow \mathcal{A}_E$ be the corresponding pull-back monomorphism of the structure rings. By virtue of Lemma 2.5 it yields a monomorphism of the DBGAs

$$\mathcal{S}^*[E'; Z'] \rightarrow \mathcal{S}^*[E; Z]. \quad (2.29)$$

Let (Y, \mathfrak{A}_F) be a simple graded manifold modelled over a vector bundle $F \rightarrow Y$. This is a graded bundle (X, Y, \mathfrak{A}_F) modelled over a composite bundle

$$F \rightarrow Y \rightarrow X. \quad (2.30)$$

The structure ring of graded functions on a simple graded manifold (Y, \mathfrak{A}_F) is the graded commutative $C^\infty(X)$ -ring $\mathcal{A}_F = \wedge F^*(Y)$ (2.17). Let the composite bundle (2.30) be provided with adapted bundle coordinates (x^λ, y^i, q^a) possessing transition functions

$$x'^\lambda(x^\mu), \quad y'^i(x^\mu, y^j), \quad q'^a = \rho_b^a(x^\mu, y^j) q^b.$$

Then the corresponding local generating basis for a simple graded manifold (Y, \mathfrak{A}_F) is (x^λ, y^i, c^a) together with transition functions

$$x'^\lambda(x^\mu), \quad y'^i(x^\mu, y^j), \quad c'^a = \rho_b^a(x^\mu, y^j) c^b.$$

We call it the local generating basis for a graded bundle (X, Y, \mathfrak{A}_F) .

2.5 Graded jet manifolds

As was mentioned above, Lagrangian theory on a smooth fibre bundle $Y \rightarrow X$ is formulated in terms of the variational bicomplex on jet manifolds J^*Y of Y . These are fibre bundles over X and, therefore, they can be regarded as trivial graded bundles $(X, J^kY, C_{J^kY}^\infty)$. Then let us describe their partners in the case of graded bundles (1.1) as follows.

Note that, given a graded manifold (X, \mathfrak{A}) and its structure ring \mathcal{A} , one can define the jet module $J^1\mathcal{A}$ of a $C^\infty(X)$ -ring \mathcal{A} [33, 64]. If (X, \mathfrak{A}_E) is a simple graded manifold modelled over a vector bundle $E \rightarrow X$, the jet module $J^1\mathcal{A}_E$ is a module of global sections of the jet bundle $J^1(\wedge E^*)$. A problem is that $J^1\mathcal{A}_E$ fails to be a structure ring of some graded manifold. By this reason, we have suggested a different construction of jets of graded manifolds, though it is applied only to simple graded manifolds [36, 65, 66].

Let (X, \mathcal{A}_E) be a simple graded manifold modelled over a vector bundle $E \rightarrow X$. Let us consider a k -order jet manifold J^kE of E . It is a vector bundle over X . Then let (X, \mathcal{A}_{J^kE}) be a simple graded manifold modelled over $J^kE \rightarrow X$. We agree to call (X, \mathcal{A}_{J^kE}) the graded k -order jet manifold of a simple graded manifold (X, \mathcal{A}_E) . Given a splitting domain $(U; x^\lambda, c^a)$ of a graded manifold (Z, \mathcal{A}_E) , we have a splitting domain

$$(U; x^\lambda, c^a, c_\lambda^a, c_{\lambda_1\lambda_2}^a, \dots, c_{\lambda_1\dots\lambda_k}^a), \quad c'_{\lambda\lambda_1\dots\lambda_r}^a = \rho_b^a(x) c_{\lambda\lambda_1\dots\lambda_r}^a + \partial_\lambda \rho_b^a(x) c_{\lambda_1\dots\lambda_r}^a,$$

of a graded jet manifold (X, \mathcal{A}_{J^kE}) .

As was mentioned above, a graded manifold is a particular graded bundle over its body (Example 2.5). Then the definition of graded jet manifolds is generalized to graded bundles over smooth manifolds as follows [68].

Let (X, Y, \mathfrak{A}_F) be a graded bundle modelled over the composite bundle (2.30). It is readily observed that the jet manifold J^rF of $F \rightarrow X$ is a vector bundle $J^rF \rightarrow J^rY$ coordinated by $(x^\lambda, y_\Lambda^i, q_\Lambda^a, 0 \leq |\Lambda| \leq r)$. Let $(J^rY, \mathfrak{A}_r = \mathfrak{A}_{J^rF})$ be a simple graded manifold modelled over this vector bundle. Its local generating basis is $(x^\lambda, y_\Lambda^i, c_\Lambda^a, 0 \leq |\Lambda| \leq r)$. We call (J^rY, \mathfrak{A}_r) the graded r -order jet manifold of a graded bundle (X, Y, \mathfrak{A}_F) .

In particular, let $Y \rightarrow X$ be a smooth bundle seen as a trivial graded bundle (X, Y, C_Y^∞) modelled over a composite bundle $Y \times \{0\} \rightarrow Y \rightarrow X$. Then its graded jet manifold is a trivial graded bundle $(X, J^rY, C_{J^rY}^\infty)$, i.e., a jet manifold J^rY of Y .

Thus, the above definition of graded jet manifolds of graded bundles is compatible with the conventional definition of jets of fibre bundles. It differs from that of jet graded bundles in [28, 51], but reproduces the heuristic notion of jets of odd ghosts in BRST field theory [3, 14].

Jet manifolds J^*Y of a fibre bundle $Y \rightarrow X$ form the inverse sequence

$$Y \xleftarrow{\pi} J^1Y \xleftarrow{\dots} J^{r-1}Y \xleftarrow{\pi_{r-1}^r} J^rY \xleftarrow{\dots}, \quad (2.31)$$

of affine bundles π_{r-1}^r . One can think of elements of its projective limit $J^\infty Y$ as being infinite order jets of sections of $Y \rightarrow X$ identified by their Taylor series at points of X . The set $J^\infty Y$ is endowed with the projective limit topology which makes $J^\infty Y$ into a paracompact Fréchet manifold [36, 71]. It is called the infinite order jet manifold. A bundle coordinate atlas (x^λ, y^i)

of Y provides $J^\infty Y$ with the adapted manifold coordinate atlas

$$\begin{aligned} (x^\lambda, y_\Lambda^i), \quad 0 \leq |\Lambda|, \quad y'_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu y'^i_\Lambda, \\ d_\lambda = \partial_\lambda + y'_\lambda \partial_i + \sum_{0 < |\Lambda|} y'_{\lambda+\Lambda} \partial_i^\Lambda, \end{aligned} \quad (2.32)$$

where d_λ are total derivatives. A fibre bundle Y is a strong deformation retract of the infinite order jet manifold $J^\infty Y$ [2, 31]. Then by virtue of the Vietoris – Begle theorem [15], there is an isomorphism

$$H^*(J^\infty Y; \mathbb{R}) = H^*(Y; \mathbb{R}) = H_{\text{DR}}^*(Y) \quad (2.33)$$

between the cohomology of $J^\infty Y$ with coefficients in the constant sheaf \mathbb{R} and the de Rham cohomology of Y .

The inverse sequence (2.31) of jet manifolds yields the direct sequence of graded differential algebras \mathcal{O}_r^* of exterior forms on finite order jet manifolds

$$\mathcal{O}^*(X) \xrightarrow{\pi^*} \mathcal{O}^*(Y) \xrightarrow{\pi_0^*} \mathcal{O}_1^* \longrightarrow \cdots \mathcal{O}_{r-1}^* \xrightarrow{\pi_{r-1}^*} \mathcal{O}_r^* \longrightarrow \cdots, \quad (2.34)$$

where π_{r-1}^* are the pull-back monomorphisms. Its direct limit

$$\mathcal{O}_\infty^* = \varinjlim \mathcal{O}_r^* \quad (2.35)$$

consists of all exterior forms on finite order jet manifolds modulo the pull-back identification. The \mathcal{O}_∞^* (2.35) is a differential graded algebra which inherits the operations of the exterior differential d and exterior product \wedge of exterior algebras \mathcal{O}_r^* . One can show that the cohomology $H^*(\mathcal{O}_\infty^*)$ of the de Rham complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}_\infty^0 \xrightarrow{d} \mathcal{O}_\infty^1 \xrightarrow{d} \cdots \quad (2.36)$$

of a differential graded algebra \mathcal{O}_∞^* equals the de Rham cohomology $H_{\text{DR}}^*(Y)$ of a fibre bundle Y [1, 36, 65]. This follows from the fact that, by virtue of the well-known theorem, the cohomology $H^*(\mathcal{O}_\infty^*)$ is isomorphic to the direct limit of the cohomology groups $H^*(\mathcal{O}_r^*) = H_{\text{DR}}^*(J^r Y)$, but all of them equal the de Rham cohomology $H_{\text{DR}}^*(Y)$ of Y because $J^r Y \rightarrow J^{r-1} Y$ are affine bundles and, consequently, Y is a strong deformation retract of any finite order jet manifold $J^r Y$.

The fibre bundles $J^{r+1} Y \rightarrow J^r Y$ (2.31) and the corresponding bundles $J^{r+1} F \rightarrow J^r F$ also yield the graded bundles

$$(J^{r+1} Y, \mathfrak{A}_{r+1}) \rightarrow (J^r Y, \mathfrak{A}_r),$$

including pull-back monomorphism of the structure rings

$$\mathcal{S}_r^0[F; Y] \rightarrow \mathcal{S}_{r+1}^0[F; Y] \quad (2.37)$$

of graded functions on graded manifolds $(J^r Y, \mathfrak{A}_r)$ and $(J^{r+1} Y, \mathfrak{A}_{r+1})$. As a consequence, we have the inverse sequence of graded manifolds

$$(Y, \mathcal{A}_F) \longleftarrow (J^1 Y, \mathfrak{A}_{J^1 F}) \longleftarrow \cdots (J^{r-1} Y, \mathfrak{A}_{J^{r-1} F}) \longleftarrow (J^r Y, \mathfrak{A}_{J^r F}) \longleftarrow \cdots$$

One can think on its inverse limit $(J^\infty Y, \mathcal{A}_{J^\infty F})$ as the graded infinite order jet manifold whose body is an infinite order jet manifold $J^\infty Y$ and whose structure sheaf $\mathcal{A}_{J^\infty F}$ is a sheaf of germs

of graded functions on graded manifolds $(J^*Y, \mathfrak{A}_{J^*F})$ [36, 66]. However $(J^\infty Y, \mathcal{A}_{J^\infty F})$ fails to be a graded manifold in a strict sense because the projective image $J^\infty Y$ of the sequence (2.31) is a Fréche manifold, but not the smooth one.

By virtue of Lemma 2.5, the differential calculus $\mathcal{S}_r^*[F; Y]$ are minimal. Therefore, the monomorphisms of structure rings (2.37) yields the pull-back monomorphisms (2.29) of DBGAs

$$\pi_r^{r+1*} : \mathcal{S}_r^*[F; Y] \rightarrow \mathcal{S}_{r+1}^*[F; Y]. \quad (2.38)$$

As a consequence, we have the direct system of DBGAs

$$\mathcal{S}^*[F; Y] \xrightarrow{\pi^*} \mathcal{S}_1^*[F; Y] \longrightarrow \cdots \mathcal{S}_{r-1}^*[F; Y] \xrightarrow{\pi_{r-1}^{r*}} \mathcal{S}_r^*[F; Y] \longrightarrow \cdots. \quad (2.39)$$

The DBGA $\mathcal{S}_\infty^*[F; Y]$ that we associate to a graded bundle (Y, \mathfrak{A}_F) is defined as the direct limit

$$\mathcal{S}_\infty^*[F; Y] = \varinjlim \mathcal{S}_r^*[F; Y] \quad (2.40)$$

of the direct system (2.39). It consists of all graded exterior forms $\phi \in \mathcal{S}^*[F; J^r Y]$ on graded manifolds $(J^r Y, \mathfrak{A}_r)$ modulo the monomorphisms (2.38). Its elements obey the relations (2.10).

The cochain monomorphisms $\mathcal{O}_r^* \rightarrow \mathcal{S}_r^*[F; Y]$ (2.23) provide a monomorphism of the direct system (2.34) to the direct system (2.39) and, consequently, a monomorphism

$$\mathcal{O}_\infty^* \rightarrow \mathcal{S}_\infty^*[F; Y] \quad (2.41)$$

of their direct limits. In particular, $\mathcal{S}_\infty^*[F; Y]$ is an \mathcal{O}_∞^0 -algebra. Accordingly, the body epimorphisms $\mathcal{S}_r^*[F; Y] \rightarrow \mathcal{O}_r^*$ yield an epimorphism of \mathcal{O}_∞^0 -algebras

$$\mathcal{S}_\infty^*[F; Y] \rightarrow \mathcal{O}_\infty^*. \quad (2.42)$$

It is readily observed that the morphisms (2.41) and (2.42) are cochain morphisms between the de Rham complex (2.36) of \mathcal{O}_∞^* and the de Rham complex

$$0 \rightarrow \mathbb{R} \longrightarrow \mathcal{S}_\infty^0[F; Y] \xrightarrow{d} \mathcal{S}_\infty^1[F; Y] \cdots \xrightarrow{d} \mathcal{S}_\infty^k[F; Y] \longrightarrow \cdots \quad (2.43)$$

of a DBGA $\mathcal{S}_\infty^*[F; Y]$. Moreover, the corresponding homomorphisms of cohomology groups of these complexes are isomorphisms as follows [36].

Lemma 2.8. *There is an isomorphism*

$$H^*(\mathcal{S}_\infty^*[F; Y]) = H_{DR}^*(Y) \quad (2.44)$$

of the cohomology $H^*(\mathcal{S}_\infty^*[F; Y])$ of the de Rham complex (2.43) to the de Rham cohomology $H_{DR}^*(Y)$ of Y .

Proof. The complex (2.43) is the direct limit of the de Rham complexes of the differential graded algebras $\mathcal{S}_r^*[F; Y]$. Therefore, the direct limit of cohomology groups of these complexes is the cohomology of the de Rham complex (2.43) in accordance with the above mentioned theorem. By virtue of the cohomology isomorphism (2.25), cohomology of the de Rham complex of $\mathcal{S}_r^*[F; Y]$ equals the de Rham cohomology of $J^r Y$ and, consequently, that of Y , which is the strong deformation retract of any jet manifold $J^r Y$. Hence, the isomorphism (2.44) holds. \square

Corollary 2.9. *Any closed graded form $\phi \in \mathcal{S}_\infty^*[F; Y]$ is decomposed into the sum $\phi = \sigma + d\xi$ where σ is a closed exterior form on Y .*

One can think of elements of $\mathcal{S}_\infty^*[F; Y]$ as being graded differential forms on an infinite order jet manifold $J^\infty Y$ [36, 66]. Indeed, let $\mathfrak{S}_r^*[F; Y]$ be the sheaf of DBGAs on $J^r Y$ and $\overline{\mathfrak{S}}_r^*[F; Y]$ its canonical presheaf. Then the above mentioned presheaf monomorphisms $\overline{\mathfrak{A}}_r \rightarrow \overline{\mathfrak{A}}_{r+1}$ yield the direct system of presheaves

$$\overline{\mathfrak{S}}^*[F; Y] \longrightarrow \overline{\mathfrak{S}}_1^*[F; Y] \longrightarrow \cdots \overline{\mathfrak{S}}_r^*[F; Y] \longrightarrow \cdots, \quad (2.45)$$

whose direct limit $\overline{\mathfrak{S}}_\infty^*[F; Y]$ is a presheaf of DBGAs on the infinite order jet manifold $J^\infty Y$. Let $\mathfrak{Q}_\infty^*[F; Y]$ be the sheaf of DBGAs of germs of the presheaf $\overline{\mathfrak{S}}_\infty^*[F; Y]$. One can think of the pair $(J^\infty Y, \mathfrak{Q}_\infty^0[F; Y])$ as being a graded Fréchet manifold, whose body is the infinite order jet manifold $J^\infty Y$ and the structure sheaf $\mathfrak{Q}_\infty^0[F; Y]$ is the sheaf of germs of graded functions on graded manifolds $(J^r Y, \mathfrak{A}_r)$. We agree to call $(J^\infty Y, \mathfrak{Q}_\infty^0[F; Y])$ the graded infinite order jet manifold. The structure module $\mathfrak{Q}_\infty^*[F; Y]$ of sections of $\mathfrak{Q}_\infty^*[F; Y]$ is a DBGA such that, given an element $\phi \in \mathfrak{Q}_\infty^*[F; Y]$ and a point $z \in J^\infty Y$, there exist an open neighbourhood U of z and a graded exterior form $\phi^{(k)}$ on some finite order jet manifold $J^k Y$ so that $\phi|_U = \pi_k^{\infty*} \phi^{(k)}|_U$. In particular, there is the monomorphism

$$\mathcal{S}_\infty^*[F; Y] \rightarrow \mathfrak{Q}_\infty^*[F; Y]. \quad (2.46)$$

Due to this monomorphism, one can restrict $\mathcal{S}_\infty^*[F; Y]$ to the coordinate chart (2.32) of $J^\infty Y$ and say that $\mathcal{S}_\infty^*[F; Y]$ as an \mathcal{O}_∞^0 -algebra is locally generated by the elements

$$(c_\Lambda^a, dx^\lambda, \theta_\Lambda^a = dc_\Lambda^a - c_{\lambda+\Lambda}^a dx^\lambda, \theta_\Lambda^i = dy_\Lambda^i - y_{\lambda+\Lambda}^i dx^\lambda), \quad 0 \leq |\Lambda|,$$

where $c_\Lambda^a, \theta_\Lambda^a$ are odd and $dx^\lambda, \theta_\Lambda^i$ are even. We agree to call (y^i, c^a) the local generating basis for $\mathcal{S}_\infty^*[F; Y]$. Let the collective symbol s^A stand for its elements. Accordingly, the notation s_Λ^A for their jets and the notation

$$\theta_\Lambda^A = ds_\Lambda^A - s_{\lambda+\Lambda}^A dx^\lambda \quad (2.47)$$

for the contact forms are introduced. For the sake of simplicity, we further denote $[A] = [s^A]$.

Remark 2.6. Let (X, Y, \mathfrak{A}_F) and $(X, Y', \mathfrak{A}_{F'})$ be graded bundles modelled over composite bundles $F \rightarrow Y \rightarrow X$ and $F' \rightarrow Y' \rightarrow X$, respectively. Let $F \rightarrow F'$ be a fibre bundle over a fibre bundle $Y \rightarrow Y'$ over X . Then we have a graded bundle

$$(X, Y, \mathfrak{A}_F) \rightarrow (X, Y', \mathfrak{A}_{F'})$$

together with the pull-back monomorphism (2.29) of DBGAs

$$\mathcal{S}^*[F'; Y'] \rightarrow \mathcal{S}^*[F; Y]. \quad (2.48)$$

Let $(X, J^r Y, \mathfrak{A}_{J^r F})$ and $(X, J^r Y', \mathfrak{A}_{J^r F'})$ be graded bundles modelled over composite bundles $J^r F \rightarrow J^r Y \rightarrow X$ and $J^r F' \rightarrow J^r Y' \rightarrow X$, respectively. Since $J^r F \rightarrow J^r F'$ is a fibre bundle over a fibre bundle $J^r Y \rightarrow J^r Y'$ over X [30], we also get a graded bundle

$$(X, J^r Y, \mathfrak{A}_{J^r F}) \rightarrow (X, J^r Y', \mathfrak{A}_{J^r F'})$$

together with the pull-back monomorphism of DBGAs

$$\mathcal{S}_r^*[F'; Y'] \rightarrow \mathcal{S}_r^*[F; Y]. \quad (2.49)$$

The monomorphisms (2.48) – (2.49), $r = 1, 2, \dots$, provide a monomorphism of the direct limits

$$\mathcal{S}_\infty^*[F'; Y'] \rightarrow \mathcal{S}_\infty^*[F; Y]. \quad (2.50)$$

of DBGAs $\mathcal{S}_r^*[F'; Y']$ and $\mathcal{S}_r^*[F; Y]$, $r = 0, 1, 2, \dots$

Remark 2.7. Let (X, Y, \mathfrak{A}_F) and $(X, Y', \mathfrak{A}_{F'})$ be graded bundles modelled over composite bundles $F \rightarrow Y \rightarrow X$ and $F' \rightarrow Y' \rightarrow X$, respectively. We define their product over X as the graded bundle

$$(X, Y, \mathfrak{A}_F) \times_X (X, Y', \mathfrak{A}_{F'}) = (X, Y \times_X Y', \mathfrak{A}_{F \times_X F'}) \quad (2.51)$$

modelled over a composite bundle

$$F \times_X F' = F \times_{Y \times Y'} F' \rightarrow Y \times_X Y' \rightarrow X. \quad (2.52)$$

Let us consider the corresponding DBGA

$$\mathcal{S}_\infty^*[F \times_X F'; Y \times_X Y']. \quad (2.53)$$

Then in accordance with Remark 2.6, there are the monomorphisms (2.50) of BGDA's

$$\mathcal{S}_\infty^*[F; Y] \rightarrow \mathcal{S}_\infty^*[F \times_X F; Y \times_X Y'], \quad \mathcal{S}_\infty^*[F'; Y'] \rightarrow \mathcal{S}_\infty^*[F \times_X F'; Y \times_X Y']. \quad (2.54)$$

3 Graded Lagrangian formalism

Let (X, Y, \mathfrak{A}_F) be a graded bundle modelled over the composite bundle (2.30) over an n -dimensional smooth manifold X , and let $\mathcal{S}_\infty^*[F; Y]$ be the associated DBGA (2.40) of graded exterior forms on graded jet manifolds of (X, Y, \mathfrak{A}_F) . As was mentioned above Grassmann-graded Lagrangian theory of even and odd variables on a graded bundle is formulated in terms of the variational bicomplex which the DBGA $\mathcal{S}_\infty^*[F; Y]$ is split in [9, 34, 36, 66].

A DBGA $\mathcal{S}_\infty^*[F; Y]$ is decomposed into $\mathcal{S}_\infty^0[F; Y]$ -modules $\mathcal{S}_\infty^{k,r}[F; Y]$ of k -contact and r -horizontal graded forms together with the corresponding projections

$$h_k : \mathcal{S}_\infty^*[F; Y] \rightarrow \mathcal{S}_\infty^{k,*}[F; Y], \quad h^m : \mathcal{S}_\infty^*[F; Y] \rightarrow \mathcal{S}_\infty^{*,m}[F; Y].$$

Accordingly, the graded exterior differential d on $\mathcal{S}_\infty^*[F; Y]$ falls into a sum $d = d_V + d_H$ of the vertical graded differential

$$d_V \circ h^m = h^m \circ d \circ h^m, \quad d_V(\phi) = \theta_\Lambda^A \wedge \partial_A^\Lambda \phi, \quad \phi \in \mathcal{S}_\infty^*[F; Y],$$

and the total graded differential

$$d_H \circ h_k = h_k \circ d \circ h_k, \quad d_H \circ h_0 = h_0 \circ d, \quad d_H(\phi) = dx^\lambda \wedge d_\lambda(\phi),$$

where

$$d_\lambda = \partial_\lambda + \sum_{0 \leq |\Lambda|} s_{\lambda+\Lambda}^A \partial_A^\Lambda$$

are the graded total derivatives. These differentials obey the nilpotent relations

$$d_H \circ d_H = 0, \quad d_V \circ d_V = 0, \quad d_H \circ d_V + d_V \circ d_H = 0.$$

A DBGA $\mathcal{S}_\infty^*[F; Y]$ also is provided with the graded projection endomorphism

$$\begin{aligned} \varrho &= \sum_{k>0} \frac{1}{k} \bar{\varrho} \circ h_k \circ h^n : \mathcal{S}_\infty^{*>0,n}[F; Y] \rightarrow \mathcal{S}_\infty^{*>0,n}[F; Y], \\ \bar{\varrho}(\phi) &= \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} \theta^A \wedge [d_\Lambda(\partial_A^\Lambda \phi)], \quad \phi \in \mathcal{S}_\infty^{>0,n}[F; Y], \end{aligned}$$

such that $\varrho \circ d_H = 0$, and with the nilpotent graded variational operator

$$\delta = \varrho \circ d : \mathcal{S}_\infty^{*,n}[F; Y] \rightarrow \mathcal{S}_\infty^{*+1,n}[F; Y]. \quad (3.1)$$

With these operators a DBGA $\mathcal{S}_\infty^*[F; Y]$ is decomposed into the Grassmann-graded variational bicomplex

$$\begin{array}{ccccccc} & \vdots & & \vdots & & \vdots & & \vdots \\ & d_V \uparrow & & d_V \uparrow & & d_V \uparrow & & -\delta \uparrow \\ 0 \rightarrow & \mathcal{S}_\infty^{1,0}[F; Y] & \xrightarrow{d_H} & \mathcal{S}_\infty^{1,1}[F; Y] & \xrightarrow{d_H} \dots & \mathcal{S}_\infty^{1,n}[F; Y] & \xrightarrow{\varrho} & \varrho(\mathcal{S}_\infty^{1,n}[F; Y]) \rightarrow 0 \\ & d_V \uparrow & & d_V \uparrow & & d_V \uparrow & & -\delta \uparrow \\ 0 \rightarrow \mathbb{R} \rightarrow & \mathcal{S}_\infty^0[F; Y] & \xrightarrow{d_H} & \mathcal{S}_\infty^{0,1}[F; Y] & \xrightarrow{d_H} \dots & \mathcal{S}_\infty^{0,n}[F; Y] & \equiv & \mathcal{S}_\infty^{0,n}[F; Y] \\ & \uparrow & & \uparrow & & \uparrow & & \\ 0 \rightarrow \mathbb{R} \rightarrow & \mathcal{O}^0(X) & \xrightarrow{d} & \mathcal{O}^1(X) & \xrightarrow{d} \dots & \mathcal{O}^n(X) & \xrightarrow{d} & 0 \\ & \uparrow & & \uparrow & & \uparrow & & \\ & 0 & & 0 & & 0 & & \end{array} \quad (3.2)$$

Its relevant cohomology has been found [36, 59, 66].

We restrict our consideration to the short variational subcomplex

$$\begin{aligned} 0 \rightarrow \mathbb{R} \rightarrow \mathcal{S}_\infty^0[F; Y] &\xrightarrow{d_H} \mathcal{S}_\infty^{0,1}[F; Y] \dots \xrightarrow{d_H} \mathcal{S}_\infty^{0,n}[F; Y] \xrightarrow{\delta} \mathbf{E}_1, \\ \mathbf{E}_1 &= \varrho(\mathcal{S}_\infty^{1,n}[F; Y]), \quad n = \dim X, \end{aligned} \quad (3.3)$$

of the bicomplex (3.2) and the subcomplex of one-contact graded forms

$$0 \rightarrow \mathcal{S}_\infty^{1,0}[F; Y] \xrightarrow{d_H} \mathcal{S}_\infty^{1,1}[F; Y] \dots \xrightarrow{d_H} \mathcal{S}_\infty^{1,n}[F; Y] \xrightarrow{\varrho} \mathbf{E}_1 \rightarrow 0. \quad (3.4)$$

They possess the following cohomology [34, 59, 66].

Theorem 3.1. *Cohomology of the complex (3.3) equals the de Rham cohomology $H_{DR}^*(Y)$ of Y .*

Theorem 3.2. *The complex (3.4) is exact.*

Decomposed into a variational bicomplex, the DBGA $\mathcal{S}_\infty^*[F; Y]$ describes Grassmann-graded Lagrangian theory on a graded bundle (X, Y, \mathfrak{A}_F) . Its graded Lagrangian is defined as an element

$$L = \mathcal{L}\omega \in \mathcal{S}_\infty^{0,n}[F; Y], \quad \omega = dx^1 \wedge \cdots \wedge dx^n, \quad (3.5)$$

of the graded variational complex (3.3). Accordingly, a graded exterior form

$$\delta L = \theta^A \wedge \mathcal{E}_A \omega = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} \theta^A \wedge d_\Lambda (\partial_A^\Lambda L) \omega \in \mathbf{E}_1 \quad (3.6)$$

is said to be its graded Euler – Lagrange operator. Its kernel defines the Euler – Lagrange equation

$$\delta L = 0, \quad \mathcal{E}_A = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} \theta^A \wedge d_\Lambda (\partial_A^\Lambda L) = 0. \quad (3.7)$$

Therefore, we agree to call a pair $(\mathcal{S}_\infty^{0,n}[F; Y], L)$ the Grassmann-graded (or, simply, graded) Lagrangian system and $\mathcal{S}_\infty^*[F; Y]$ its structure algebra.

The following is a corollary of Theorems 3.1 and (3.2 [34, 36, 66]).

Theorem 3.3. *Every d_H -closed graded form $\phi \in \mathcal{S}_\infty^{0,m<n}[F; Y]$ falls into the sum*

$$\phi = h_0 \sigma + d_H \xi, \quad \xi \in \mathcal{S}_\infty^{0,m-1}[F; Y], \quad (3.8)$$

where σ is a closed m -form on Y . Any δ -closed (i.e., variationally trivial) graded Lagrangian $L \in \mathcal{S}_\infty^{0,n}[F; Y]$ is a sum

$$L = h_0 \sigma + d_H \xi, \quad \xi \in \mathcal{S}_\infty^{0,n-1}[F; Y], \quad (3.9)$$

where σ is a closed n -form on Y .

Proof. The complex (3.3) possesses the same cohomology as the short variational complex

$$0 \rightarrow \mathbb{R} \rightarrow \mathcal{O}_\infty^0 \xrightarrow{d_H} \mathcal{O}_\infty^{0,1} \cdots \xrightarrow{d_H} \mathcal{O}_\infty^{0,n} \xrightarrow{\delta} \mathbf{E}_1 \quad (3.10)$$

of the differential graded algebra \mathcal{O}_∞^* . The monomorphism (2.41) and the body epimorphism (2.42) yield the corresponding cochain morphisms of the complexes (3.3) and (3.10). Therefore, cohomology of the complex (3.3) is the image of the cohomology of \mathcal{O}_∞^* . \square

Corollary 3.4. *Any variationally trivial odd Lagrangian is d_H -exact.*

The exactness of the complex (3.4) at the term $\mathcal{S}_\infty^{1,n}[F; Y]$ results in the following [34, 36, 66].

Theorem 3.5. *Given a graded Lagrangian L , there is the decomposition*

$$dL = \delta L - d_H \Xi_L, \quad \Xi \in \mathcal{S}_\infty^{n-1}[F; Y], \quad (3.11)$$

$$\Xi_L = L + \sum_{s=0} \theta_{\nu_s \dots \nu_1}^A \wedge F_A^{\lambda \nu_s \dots \nu_1} \omega_\lambda, \quad (3.12)$$

$$F_A^{\nu_k \dots \nu_1} = \partial_A^{\nu_k \dots \nu_1} \mathcal{L} - d_\lambda F_A^{\lambda \nu_k \dots \nu_1} + \sigma_A^{\nu_k \dots \nu_1}, \quad k = 1, 2, \dots,$$

where local graded functions σ obey the relations

$$\sigma_A^\nu = 0, \quad \sigma_A^{(\nu_k \nu_{k-1}) \dots \nu_1} = 0.$$

The form Ξ_L (3.12) provides a global Lepage equivalent of a graded Lagrangian L . In particular, one can locally choose Ξ_L (3.12) where all functions σ vanish.

The formula (3.11), called the variational decomposition, play a prominent role in the formulation and the proof of Noether theorems.

Example 3.1. Let us consider first order Lagrangian theory on a fibre bundle $Y \rightarrow X$. Its structure algebra (2.40) is the \mathcal{O}_∞^* (2.35). The first order Lagrangian (3.5) reads

$$L = \mathcal{L}\omega : J^1Y \rightarrow \bigwedge^n T^*X. \quad (3.13)$$

The corresponding second-order Euler – Lagrange operator (3.6) takes a form

$$\begin{aligned} \mathcal{E}_L : J^2Y &\rightarrow T^*Y \wedge \left(\bigwedge^n T^*X\right), \\ \mathcal{E}_L &= (\partial_i \mathcal{L} - d_\lambda \pi_i^\lambda) \theta^i \wedge \omega, \quad \pi_i^\lambda = \partial_i^\lambda \mathcal{L}. \end{aligned} \quad (3.14)$$

Its kernel defines the second order Euler – Lagrange equation

$$(\partial_i - d_\lambda \partial_i^\lambda) \mathcal{L} = 0. \quad (3.15)$$

Given the first order Lagrangian L (3.13), its Lepage equivalents Ξ_L (3.12) in the variational decomposition (3.11) read

$$\Xi_L = L + (\pi_i^\lambda - d_\mu \sigma_i^{\mu\lambda}) \theta^i \wedge \omega_\lambda + \sigma_i^{\lambda\mu} \theta_\mu^i \wedge \omega_\lambda, \quad (3.16)$$

where $\sigma_i^{\mu\lambda} = -\sigma_i^{\lambda\mu}$ are skew-symmetric local functions on Y . One usually choose the Poincaré – Cartan form

$$H_L = \mathcal{L}\omega + \pi_i^\lambda \theta^i \wedge \omega_\lambda. \quad (3.17)$$

4 First Noether theorem

As was mentioned above, the first Noether theorem (Theorem 4.7) is a straightforward corollary of the variational decomposition (3.11).

4.1 Infinitesimal graded transformations of Lagrangian systems

Given a graded Lagrangian system $(\mathcal{S}_\infty^*[F; Y], L)$, by its infinitesimal transformations are meant contact graded derivations of the real graded commutative ring $\mathcal{S}_\infty^0[F; Y]$. These derivations constitute a $\mathcal{S}_\infty^0[F; Y]$ -module $\mathfrak{d}\mathcal{S}_\infty^0[F; Y]$ which is a real Lie superalgebra with respect to the Lie superbracket (2.6). The following holds [34, 36].

Theorem 4.1. *The derivation module $\mathfrak{d}\mathcal{S}_\infty^0[F; Y]$ is isomorphic to the $\mathcal{S}_\infty^0[F; Y]$ -dual $(\mathcal{S}_\infty^1[F; Y])^*$ of the module of graded one-forms $\mathcal{S}_\infty^1[F; Y]$.*

Proof. At first, let us show that $\mathcal{S}_\infty^*[F; Y]$ is generated by elements df , $f \in \mathcal{S}_\infty^0[F; Y]$. It suffices to justify that any element of $\mathcal{S}_\infty^1[F; Y]$ is a finite $\mathcal{S}_\infty^0[F; Y]$ -linear combination of elements df , $f \in \mathcal{S}_\infty^0[F; Y]$. Indeed, every $\phi \in \mathcal{S}_\infty^1[F; Y]$ is a graded exterior form on some finite order jet manifold J^rY , i.e., a section of a vector bundle $\overline{\mathcal{V}}_{J^rF} \rightarrow J^rY$ in accordance

with Lemma 2.4. Then by virtue of the classical Serre – Swan theorem, a $C^\infty(J^r Y)$ -module $\mathcal{S}_r^1[F; Y]$ of graded one-forms on $J^r Y$ is a projective module of finite rank, i.e., ϕ is represented by a finite $C^\infty(J^r Y)$ -linear combination of elements df , $f \in \mathcal{S}_r^0[F; Y] \subset \mathcal{S}_\infty^0[F; Y]$. Any element $\Phi \in (\mathcal{S}_\infty^1[F; Y])^*$ yields a derivation $\vartheta_\Phi(f) = \Phi(df)$ of the real ring $\mathcal{S}_\infty^0[F; Y]$. Since the module $\mathcal{S}_\infty^1[F; Y]$ is generated by elements df , $f \in \mathcal{S}_\infty^0[F; Y]$, different elements of $(\mathcal{S}_\infty^1[F; Y])^*$ provide different derivations of $\mathcal{S}_\infty^0[F; Y]$, i.e., there is a monomorphism $(\mathcal{S}_\infty^0[F; Y])^* \rightarrow \mathfrak{d}\mathcal{S}_\infty^0[F; Y]$. By the same formula, any derivation $\vartheta \in \mathfrak{d}\mathcal{S}_\infty^0[F; Y]$ sends $df \rightarrow \vartheta(f)$ and, since $\mathcal{S}_\infty^0[F; Y]$ is generated by elements df , it defines a morphism $\Phi_\vartheta : \mathcal{S}_\infty^1[F; Y] \rightarrow \mathcal{S}_\infty^0[F; Y]$. Moreover, different derivations ϑ provide different morphisms Φ_ϑ . Thus, we have a monomorphism $\mathfrak{d}\mathcal{S}_\infty^0[F; Y] \rightarrow (\mathcal{S}_\infty^1[F; Y])^*$ and, consequently, isomorphism $\mathfrak{d}\mathcal{S}_\infty^0[F; Y] = (\mathcal{S}_\infty^0[F; Y])^*$. \square

The proof of Theorem 4.1 gives something more. It follows that the DBGA $\mathcal{S}_\infty^*[F; Y]$ is minimal differential calculus over the real graded commutative ring $\mathcal{S}_\infty^0[F; Y]$.

Let $\vartheta \rfloor \phi$, $\vartheta \in \mathfrak{d}\mathcal{S}_\infty^0[F; Y]$, $\phi \in \mathcal{S}_\infty^1[F; Y]$, denote the corresponding interior product. Extended to the DBGA $\mathcal{S}_\infty^*[F; Y]$, it obeys the rule

$$\vartheta \rfloor (\phi \wedge \sigma) = (\vartheta \rfloor \phi) \wedge \sigma + (-1)^{|\phi|+[\phi][\vartheta]} \phi \wedge (\vartheta \rfloor \sigma), \quad \phi, \sigma \in \mathcal{S}_\infty^*[F; Y].$$

Restricted to a coordinate chart (2.32) of $J^\infty Y$, the algebra $\mathcal{S}_\infty^*[F; Y]$ is a free $\mathcal{S}_\infty^0[F; Y]$ -module generated by one-forms dx^λ , θ_Λ^A . Due to the isomorphism stated in Theorem 4.1, any graded derivation $\vartheta \in \mathfrak{d}\mathcal{S}_\infty^0[F; Y]$ takes a local form

$$\vartheta = \vartheta^\lambda \partial_\lambda + \vartheta^A \partial_A + \sum_{0 < |\Lambda|} \vartheta_\Lambda^A \partial_A^\Lambda, \quad (4.1)$$

where

$$\partial_A^\Lambda (s_\Sigma^B) = \partial_A^\Lambda \rfloor ds_\Sigma^B = \delta_A^B \delta_\Sigma^\Lambda$$

up to permutations of multi-indices Λ and Σ . Its coefficients ϑ^λ , ϑ^A , ϑ_Λ^A are local smooth functions of finite jet order possessing the transformation law

$$\begin{aligned} \vartheta'^\lambda &= \frac{\partial x'^\lambda}{\partial x^\mu} \vartheta^\mu, & \vartheta'^A &= \frac{\partial s'^A}{\partial s^B} \vartheta^B + \frac{\partial s'^A}{\partial x^\mu} \vartheta^\mu, \\ \vartheta'_\Lambda^A &= \sum_{|\Sigma| \leq |\Lambda|} \frac{\partial s'_\Lambda^A}{\partial s_\Sigma^B} \vartheta_\Sigma^B + \frac{\partial s'_\Lambda^A}{\partial x^\mu} \vartheta^\mu. \end{aligned} \quad (4.2)$$

Every graded derivation ϑ (4.1) of a ring $\mathcal{S}_\infty^0[F; Y]$ yields a derivation (called the Lie derivative) \mathbf{L}_ϑ of the BGDA $\mathcal{S}_\infty^*[F; Y]$ given by the relations

$$\begin{aligned} \mathbf{L}_\vartheta \phi &= \vartheta \rfloor d\phi + d(\vartheta \rfloor \phi), & \phi &\in \mathcal{S}_\infty^*[F; Y], \\ \mathbf{L}_\vartheta (\phi \wedge \sigma) &= \mathbf{L}_\vartheta (\phi) \wedge \sigma + (-1)^{[\vartheta][\phi]} \phi \wedge \mathbf{L}_\vartheta (\sigma), \end{aligned}$$

of the DBGA $\mathcal{S}_\infty^*[F; Y]$.

The graded derivation ϑ (4.1) is called contact if the Lie derivative \mathbf{L}_ϑ preserves the ideal of contact graded forms of the DBGA $\mathcal{S}_\infty^*[F; Y]$ generated by the contact one-forms (2.47).

Lemma 4.2. *With respect to the local generating basis (s^A) for the DBGA $\mathcal{S}_\infty^*[F; Y]$, any its contact graded derivation takes a form*

$$\vartheta = \vartheta_H + \vartheta_V = v^\lambda d_\lambda + [v^A \partial_A + \sum_{|\Lambda| > 0} d_\Lambda (v^A - s_\mu^A v^\mu) \partial_A^\Lambda], \quad (4.3)$$

where ϑ_H and ϑ_V denotes the horizontal and vertical parts of ϑ [34, 36].

Proof. The expression (4.3) results from a direct computation similar to that of the first part of Bäcklund's theorem [42]. One can then justify that local functions (4.3) satisfy the transformation law (4.2). \square

A glance at the expression (4.3) shows that a contact graded derivation ϑ is the infinite order jet prolongation

$$\vartheta = J^\infty v \quad (4.4)$$

of its restriction

$$v = v^\lambda \partial_\lambda + v^A \partial_A = v_H + v_V = v^\lambda d_\lambda + (u^A \partial_A - s_\lambda^A \partial_\lambda^A) \quad (4.5)$$

to the graded commutative ring $S^0[F; Y]$. We call the v (4.5) the generalized graded vector field on a graded manifold (Y, \mathfrak{A}_F) . This fails to be a graded vector field on (Y, \mathfrak{A}_F) because its component depends on jets of elements of the local generating basis for (Y, \mathfrak{A}_F) in general. At the same time, any graded vector field u on (Y, \mathfrak{A}_F) is the generalized graded vector field (4.5) generating the contact graded derivation $J^\infty u$ (4.4).

In particular, the vertical contact graded derivation (4.5) reads

$$\vartheta = v^A \partial_A + \sum_{|\Lambda| > 0} d_\Lambda v^A \partial_A^\Lambda. \quad (4.6)$$

Lemma 4.3. *Any vertical contact graded derivation (4.6) obeys the relations*

$$\vartheta \rfloor d_H \phi = -d_H(\vartheta \rfloor \phi), \quad (4.7)$$

$$\mathbf{L}_\vartheta(d_H \phi) = d_H(\mathbf{L}_\vartheta \phi), \quad \phi \in \mathcal{S}_\infty^*[F; Y]. \quad (4.8)$$

Proof. It is easily justified that, if ϕ and ϕ' satisfy the relation (4.7), then $\phi \wedge \phi'$ does well. Then it suffices to prove the relation (4.7) when ϕ is a function and $\phi = \theta_\Lambda^A$. The result follows from the equalities

$$\vartheta \rfloor \theta_\Lambda^A = v_\Lambda^A, \quad d_H(v_\Lambda^A) = v_{\Lambda+\Lambda}^A dx^\lambda, \quad d_H \theta_\Lambda^A = dx^\lambda \wedge \theta_{\Lambda+\Lambda}^A, \quad (4.9)$$

$$d_\lambda \circ v_\Lambda^A \partial_A^\Lambda = v_\Lambda^A \partial_A^\Lambda \circ d_\lambda. \quad (4.10)$$

The relation (4.8) is a corollary of the equality (4.7). \square

The vertical contact graded derivation ϑ (4.6) is said to be nilpotent if

$$\begin{aligned} \mathbf{L}_\vartheta(\mathbf{L}_\vartheta \phi) &= \sum_{0 \leq |\Sigma|, 0 \leq |\Lambda|} (v_\Sigma^B \partial_B^\Sigma (v_\Lambda^A) \partial_A^\Lambda + \\ &(-1)^{[s^B][v^A]} v_\Sigma^B v_\Lambda^A \partial_B^\Sigma \partial_A^\Lambda) \phi = 0 \end{aligned} \quad (4.11)$$

for any horizontal graded form $\phi \in S_\infty^{0,*}$.

Lemma 4.4. *The vertical contact graded derivation (4.6) is nilpotent only if it is odd and iff the equality*

$$\mathbf{L}_\vartheta(v^A) = \sum_{0 \leq |\Sigma|} v_\Sigma^B \partial_B^\Sigma (v^A) = 0$$

holds for all v^A .

Proof. There is the relation

$$d_\lambda \circ v_\Lambda^A \partial_A^\lambda = v_\Lambda^A \partial_A^\lambda \circ d_\lambda. \quad (4.12)$$

Then the result follows from the equality (4.11) where one puts $\phi = s^A$ and $\phi = s_\Lambda^A s_\Sigma^B$. \square

Remark 4.1. If there is no danger of confusion, the common symbol v further stands for a generalized graded vector field v , the contact graded derivation ϑ (4.4) determined by v , and the Lie derivative \mathbf{L}_ϑ . We agree to call all these operators, simply, a graded derivation of the structure algebra of a Lagrangian system.

Remark 4.2. For the sake of convenience, right graded derivations

$$\overleftarrow{v} = \overleftarrow{\partial}_A v^A \quad (4.13)$$

also are considered. They act on graded functions and differential forms ϕ on the right by the rules

$$\begin{aligned} \overleftarrow{v}(\phi) &= d\phi[\overleftarrow{v}] + \phi[\overleftarrow{v}], \\ \overleftarrow{v}(\phi \wedge \phi') &= (-1)^{[\phi']} \overleftarrow{v}(\phi) \wedge \phi' + \phi \wedge \overleftarrow{v}(\phi'), \\ \theta_{\Lambda A}[\overleftarrow{\partial}^{\Sigma B}] &= \delta_B^A \delta_\Lambda^\Sigma. \end{aligned}$$

4.2 Lagrangian symmetries and conservation laws

Let $(\mathcal{S}_\infty^*[F; Y], L)$ be a Grassmann-graded Lagrangian system. A contact graded derivation ϑ (4.3) is called the variational symmetry of a graded Lagrangian L if a Lie derivative $\mathbf{L}_\vartheta L$ of L along ϑ is d_H -exact, i.e.,

$$\mathbf{L}_\vartheta L = d_H \sigma. \quad (4.14)$$

A corollary of the variational decomposition (3.11) is the first variational formula for a graded Lagrangian [6, 34, 36].

Theorem 4.5. *The Lie derivative of a graded Lagrangian along any contact graded derivation (4.3) obeys the first variational formula*

$$\mathbf{L}_\vartheta L = v_V \rfloor \delta L + d_H(h_0(\vartheta \rfloor \Xi_L)) + d_V(v_H \rfloor \omega) \mathcal{L}, \quad (4.15)$$

where Ξ_L is the Lepage equivalent (3.12) of a Lagrangian L .

Proof. The formula (4.15) comes from the decomposition (3.11) and the relations (4.7) – (4.8) as follows:

$$\begin{aligned} \mathbf{L}_\vartheta L &= \vartheta \rfloor dL + d(\vartheta \rfloor L) = [\vartheta_V \rfloor dL - d_V \mathcal{L} \wedge v_H \rfloor \omega] + [d_H(v_H \rfloor L) + \\ &\quad d_V(\mathcal{L} v_H \rfloor \omega)] = \vartheta_V \rfloor dL + d_H(v_H \rfloor L) + d_V(v_H \rfloor \omega) \mathcal{L} = \\ &\quad v_V \rfloor \delta L - \vartheta_V \rfloor d_H \Xi_L + d_H(v_H \rfloor L) + d_V(v_H \rfloor \omega) \mathcal{L} = \\ &\quad v_V \rfloor \delta L + d_H(\vartheta_V \rfloor \Xi_L + v_H \rfloor L) + d_V(v_H \rfloor \omega) \mathcal{L}, \end{aligned}$$

where

$$\vartheta_V \rfloor \Xi_L = h_0(v_V \rfloor \Xi_L)$$

since $\Xi_L - L$ is a one-contact form and

$$\vartheta_H \rfloor L = h_0(v_H \rfloor \Xi_L).$$

□

A glance at the expression (4.15) shows the following.

Lemma 4.6. (i) *A contact graded derivation ϑ is a variational symmetry only if it is projected onto X .*

(ii) *Any projectable contact graded derivation is a variational symmetry of a variationally trivial graded Lagrangian. It follows that, if ϑ is a variational symmetry of a graded Lagrangian L , it also is a variational symmetry of a Lagrangian $L + L_0$, where L_0 is a variationally trivial graded Lagrangian.*

(iii) *A contact graded derivations ϑ is a variational symmetry iff its vertical part v_V (4.3) is well.*

(iv) *It is a variational symmetry iff the graded density $v_V \rfloor \delta L$ is d_H -exact.*

Variational symmetries of a graded Lagrangian L constitute a real vector subspace \mathcal{G}_L of the graded derivation module $\mathfrak{dS}_\infty^0[F; Y]$. By virtue of item (ii) of Lemma 4.6, the Lie superbracket

$$\mathbf{L}_{[\vartheta, \vartheta']} = [\mathbf{L}_\vartheta, \mathbf{L}_{\vartheta'}]$$

of variational symmetries is a variational symmetry and, therefore, their vector space \mathcal{G}_L is a real Lie superalgebra.

An immediate corollary of the first variational formula (4.15) is the following first Noether theorem.

Theorem 4.7. *If the contact graded derivation ϑ (4.3) is a variational symmetry of a graded Lagrangian L , the first variational formula (4.15) leads to the weak conservation law*

$$0 \approx d_H(h_0(\vartheta \rfloor \Xi_L) - \sigma) \quad (4.16)$$

of the current

$$\mathcal{J}_\vartheta = \mathcal{J}_\vartheta^\mu \omega_\mu = \sigma - h_0(\vartheta \rfloor \Xi_L), \quad \omega_\mu = \partial_\mu \rfloor \omega. \quad (4.17)$$

on the shell $\text{Ker} \delta L$ (3.7).

Obviously, the conserved current (4.17) is defined up to a d_H -closed graded horizontal $(n-1)$ -form, e.g. a total differential $d_H U$ of some graded horizontal $(n-2)$ -form

$$U = \frac{1}{2} U^{\nu\mu} \omega_{\nu\mu}, \quad \omega_{\nu\mu} = \partial_\nu \rfloor \omega_\mu, \quad (4.18)$$

called the superpotential.

A variational symmetry ϑ of a graded Lagrangian L is called its exact symmetry or, simply, a symmetry if

$$\mathbf{L}_\vartheta L = 0. \quad (4.19)$$

In this case, the weak conservation law (4.16) takes a form

$$0 \approx d_H(h_0(\vartheta)\Xi_L) = -d_H\mathcal{J}_\vartheta, \quad (4.20)$$

where

$$\mathcal{J}_\vartheta = \mathcal{J}_\vartheta^\mu \omega_\mu = -h_0(\vartheta)\Xi_L \quad (4.21)$$

is called the symmetry current. Of course, the symmetry current (4.21) also is defined with the accuracy of a d_H -closed term.

Let ϑ be an exact symmetry of a Lagrangian L . Whenever L_0 is a variationally trivial Lagrangian, ϑ is a variational symmetry of the Lagrangian $L+L_0$ such that the weak conservation law (4.16) for this Lagrangian is reduced to the weak conservation law (4.20) for a Lagrangian L as follows:

$$\mathbf{L}_\vartheta(L+L_0) = d_H\sigma \approx d_H\sigma - d_H\mathcal{J}_\vartheta.$$

Remark 4.3. In accordance with the standard terminology, variational and exact symmetries generated by generalized graded vector fields (4.5) are called generalized symmetries because they depend on derivatives of variables. Generalized symmetries of differential equations and Lagrangian systems have been intensively investigated [17, 22, 34, 42, 48, 52]. Accordingly, by variational symmetries and symmetries one means only those generated by vector fields v on a graded bundle (Y, \mathfrak{A}_F) . We agree to call them classical symmetries. In this case, the relation

$$\mathbf{L}_\vartheta\mathcal{E}_L = \delta(\mathbf{L}_\vartheta L) = 0 \quad (4.22)$$

holds [30, 52]. It follows that ϑ also is a symmetry of the Euler – Lagrange operator \mathcal{E}_L of L . However, the equality (4.22) fails to be true in the case of generalized symmetries.

Example 4.4. Following Example 3.1, let us consider first order Lagrangian theory on a fibre bundle $Y \rightarrow X$. Given its Lagrangian L (3.13), let v (4.5) be its classical symmetry given by a projectable vector field

$$v = u = u^\lambda \partial_\lambda + u^i \partial_i = u_H + u_V = u^\lambda d_\lambda + (u^i \partial_i - y_\lambda^i \partial_i^\lambda)$$

on a fibre bundle $Y \rightarrow X$. In this case, it is sufficient to consider the first order jet prolongation

$$J^1 u = u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y_\mu^i d_\lambda u^\mu) \partial_i^\lambda. \quad (4.23)$$

of u onto $J^1 Y$, but not the infinite order one (4.4). Then the first variational formula (4.15) takes a form

$$\mathbf{L}_{J^1 u} L = u_V \mathcal{E}_L + d_H(h_0(u)H_L), \quad (4.24)$$

where $\Xi_L = H_L$ is the Poincaré – Cartan form (3.17). Its coordinate expression reads

$$\begin{aligned} \partial_\lambda u^\lambda \mathcal{L} + [u^\lambda \partial_\lambda + u^i \partial_i + (d_\lambda u^i - y_\mu^i \partial_\lambda u^\mu) \partial_i^\lambda] \mathcal{L} = \\ (u^i - y_\lambda^i u^\lambda) \mathcal{E}_i - d_\lambda [\pi_i^\lambda (u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}]. \end{aligned} \quad (4.25)$$

If u is an exact symmetry of L , we obtain the weak conservation law (4.20):

$$0 \approx -d_\lambda [\pi_i^\lambda (u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}], \quad (4.26)$$

of the symmetry current (4.21):

$$\mathcal{J}_u = [\pi_i^\lambda (u^\mu y_\mu^i - u^i) - u^\lambda \mathcal{L}] \omega_\lambda \quad (4.27)$$

along a vector field u .

4.3 Gauge symmetries

Treating gauge symmetries of Lagrangian field theory, one is traditionally based on an example of Yang – Mills gauge theory of principal connections on principal bundles (Section 6.1). This notion of gauge symmetries has been generalized to Lagrangian theory on an arbitrary fibre bundle $Y \rightarrow X$ [7, 35, 36]. Here, we extend this notion to Lagrangian theory on graded bundles in a general setting (Definition 4.8).

Let $(\mathcal{S}_\infty^*[F; Y], L)$ be a Grassmann-graded Lagrangian system on a graded bundle (X, Y, \mathfrak{A}_F) with the local generating basis (s^A) . Let

$$E = E^0 \oplus_X E^1$$

be a graded vector bundle over X possessing an even part $E^0 \rightarrow X$ and the odd one $E^1 \rightarrow X$. We regard it as a composite bundle

$$E \rightarrow E^0 \rightarrow X \quad (4.28)$$

and consider a graded bundle (X, E^0, \mathfrak{A}_E) modelled over this composite bundle. Then we define the product (2.51) of graded bundles (X, Y, \mathfrak{A}_F) and (X, E^0, \mathfrak{A}_E) over the product (2.52) of the composite bundles E (4.28) and F (2.30). It reads

$$(X, E^0 \times_X Y, \mathfrak{A}_{E \times_X F}) \quad (4.29)$$

Let us consider the corresponding DBGA

$$\mathcal{S}_\infty^*[E \times_X F; E^0 \times_X Y] \quad (4.30)$$

together with the monomorphisms (2.54) of DBGAs

$$\mathcal{S}_\infty^*[F; Y] \rightarrow \mathcal{S}_\infty^*[E \times_X F; E^0 \times_X Y], \quad \mathcal{S}_\infty^*[E; E^0] \rightarrow \mathcal{S}_\infty^*[E \times_X F; E^0 \times_X Y] \quad (4.31)$$

Given a Lagrangian $L \in \mathcal{S}_\infty^{0,n}[F; Y]$, let us define its pull-back

$$L \in \mathcal{S}_\infty^{0,n}[F; Y] \subset \mathcal{S}_\infty^*[E \times_X F; E^0 \times_X Y], \quad (4.32)$$

and consider an extended Lagrangian system

$$(\mathcal{S}_\infty^*[E \times_X F; E^0 \times_X Y], L) \quad (4.33)$$

provided with the local generating basis (s^A, c^r) .

Definition 4.8. A gauge transformation of the Lagrangian L (4.32) is defined to be the contact graded derivation ϑ (4.4) of the ring $\mathcal{S}_\infty^0[E \times_X F; E^0 \times_X Y]$ (4.30) such that a derivation ϑ equals zero on a subring

$$\mathcal{S}_\infty^0[E; E^0] \subset \mathcal{S}_\infty^0[E \times_X F; E^0 \times_X Y],$$

A gauge transformation ϑ is called the gauge symmetry if it is a variational symmetry of the Lagrangian L (4.32), i.e., a density $v_V \rfloor \delta L$ is d_H -exact.

In view of the first condition in Definition 4.8, the variables c^r of the extended Lagrangian system (4.33) can be treated as gauge parameters of a gauge symmetry ϑ .

Furthermore, we additionally assume that a gauge symmetry ϑ is linear in gauge parameters c^r and their jets c_Λ^r (see Remark 5.10). Then the generalized graded vector field v (4.5) generating a gauge symmetry ϑ reads

$$v = \left(\sum_{0 \leq |\Lambda| \leq m} v_r^{\lambda\Lambda}(x^\mu) c_\Lambda^r \right) \partial_\lambda + \left(\sum_{0 \leq |\Lambda| \leq m} v_r^{A\Lambda}(x^\mu, s_\Sigma^B) c_\Lambda^r \right) \partial_A. \quad (4.34)$$

In accordance with Remark 4.1, we also call it the gauge symmetry.

By virtue of item (iii) of Lemma 4.6, the generalized vector field v (4.34) is a gauge symmetry iff its vertical part is so. Therefore, we can restrict our consideration to vertical gauge symmetries.

Remark 4.5. Let $E = E^0$, i.e., gauge parameters are even. A glance at the expression (4.34) shows that, in this case, a gauge symmetry v is a linear differential operator on a real space of sections of a vector bundle $E^0 \rightarrow X$ with values in a real space \mathcal{G}_L of variational symmetries of a Lagrangian L [7, 6, 36].

4.4 Gauge conservation laws

Being a variational symmetry, the gauge symmetry v (4.34) defines the weak conservation law (4.16). The peculiarity of this conservation law is that the conserved current \mathcal{J}_v (4.17) is reduced to the superpotential (4.18) as follows.

Theorem 4.9. If v (4.34) is a gauge symmetry of a Lagrangian L , the corresponding conserved current \mathcal{J}_v (4.17) is linear in gauge parameters (up to a d_h -closed term), and it takes a form

$$\mathcal{J}_v = W + d_H U = (W^\mu + d_\nu U^{\nu\mu}) \omega_\mu, \quad (4.35)$$

where a term W vanishes on-shell and $U^{\nu\mu} = -U^{\mu\nu}$ is the superpotential (4.18) which takes the form (4.43).

Proof. Let the gauge symmetry v (4.34) be at most of jet order N in parameters. Then the conserved current \mathcal{J}_v (4.17), being linear in gauge parameters, is decomposed into a sum

$$\begin{aligned} \mathcal{J}_v^\mu &= J_r^{\mu\mu_1 \dots \mu_M} c_{\mu_1 \dots \mu_M}^r + \sum_{1 \leq k \leq M} J_r^{\mu\mu_k \dots \mu_M} c_{\mu_k \dots \mu_M}^r + \\ &J_r^{\mu\mu_M} c_{\mu_M}^r + J_r^\mu c^r, \quad N \leq M, \end{aligned} \quad (4.36)$$

and the first variational formula (4.15) takes a form

$$\begin{aligned} 0 &= \left[\sum_{k=1}^N v_{Vr}^{A\mu_k \dots \mu_N} c_{\mu_k \dots \mu_N}^r + v_{Vr}^A c^r \right] \mathcal{E}_A - \\ &d_\mu \left(\sum_{k=1}^M J_r^{\mu\mu_k \dots \mu_M} c_{\mu_k \dots \mu_M}^r + J_r^\mu c^r \right). \end{aligned}$$

This equality provides the following set of equalities for each $c_{\mu_1 \dots \mu_M}^r$, $c_{\mu_k \dots \mu_M}^r$ ($k = 1, \dots, M - N - 1$), $c_{\mu_k \dots \mu_N}^r$ ($k = 1, \dots, N - 1$), c_{μ}^r and c^r :

$$0 = J_r^{(\mu\mu_1) \dots \mu_M}, \quad (4.37)$$

$$0 = J_r^{(\mu_k \mu_{k+1}) \dots \mu_M} + d_\nu J_r^{\nu \mu_k \dots \mu_M}, \quad 1 \leq k < M - N, \quad (4.38)$$

$$0 = v_{V_r}^{A \mu_k \dots \mu_N} \mathcal{E}_A - J_r^{(\mu_k \mu_{k+1}) \dots \mu_N} - d_\nu J_r^{\nu \mu_k \dots \mu_N}, \quad 1 \leq k < N, \quad (4.39)$$

$$0 = v_{V_r}^{A \mu} \mathcal{E}_A - J_r^\mu - d_\nu J_r^{\nu \mu}, \quad (4.40)$$

where $(\mu\nu)$ means symmetrization of indices in accordance with the splitting

$$J_r^{\mu_k \mu_{k+1} \dots \mu_N} = J_r^{(\mu_k \mu_{k+1}) \dots \mu_N} + J_r^{[\mu_k \mu_{k+1}] \dots \mu_N}.$$

We also have the equality

$$0 = v_{V_r}^A \mathcal{E}_A - d_\mu J_r^\mu, \quad (4.41)$$

With the equalities (4.37) – (4.40), the decomposition (4.36) takes a form

$$\begin{aligned} \mathcal{J}_v^\mu &= J_r^{[\mu\mu_1] \dots \mu_M} c_{\mu_1 \dots \mu_M}^r + \\ &\quad \sum_{1 < k \leq M-N} [(J_r^{[\mu\mu_k] \dots \mu_M} - d_\nu J_r^{\nu \mu_k \dots \mu_M}) c_{\mu_k \dots \mu_M}^r] + \\ &\quad \sum_{1 < k < N} [(v_{V_r}^i A_r^{\mu\mu_k \dots \mu_N} \mathcal{E}_A - d_\nu J_r^{\nu \mu_k \dots \mu_N} + J_r^{[\mu\mu_k] \dots \mu_N}) c_{\mu_k \dots \mu_N}^r] + \\ &\quad (v_{V_r}^{A \mu\mu_N} \mathcal{E}_A - d_\nu J_r^{\nu \mu\mu_N} + J_r^{[\mu\mu_N]}) c_{\mu_N}^r + (v_{V_r}^A \mathcal{E}_A - d_\nu J_r^{\nu \mu}) c^r. \end{aligned}$$

A direct computation

$$\begin{aligned} \mathcal{J}_v^\mu &= d_\nu (J_r^{[\mu\nu] \mu_2 \dots \mu_M} c_{\mu_2 \dots \mu_M}^r) - d_\nu J_r^{[\mu\nu] \mu_2 \dots \mu_M} c_{\mu_2 \dots \mu_M}^r + \\ &\quad \sum_{1 < k \leq M-N} [d_\nu (J_r^{[\mu\nu] \mu_{k+1} \dots \mu_M} c_{\mu_{k+1} \dots \mu_M}^r) - \\ &\quad d_\nu J_r^{[\mu\nu] \mu_{k+1} \dots \mu_M} c_{\mu_{k+1} \dots \mu_M}^r - d_\nu J_r^{\nu \mu_{k+1} \dots \mu_M} c_{\mu_{k+1} \dots \mu_M}^r] + \\ &\quad \sum_{1 < k < N} [(v_{V_r}^A \mu_{\mu_k \dots \mu_N} \mathcal{E}_A - d_\nu J_r^{\nu \mu_k \dots \mu_N}) c_{\mu_k \dots \mu_N}^r + \\ &\quad d_\nu (J_r^{[\mu\nu] \mu_{k+1} \dots \mu_N} c_{\mu_{k+1} \dots \mu_N}^r) - d_\nu J_r^{[\mu\nu] \mu_{k+1} \dots \mu_N} c_{\mu_{k+1} \dots \mu_N}^r] + \\ &\quad [(v_{V_r}^{A \mu\mu_N} \mathcal{E}_A - d_\nu J_r^{\nu \mu\mu_N}) c_{\mu_N}^r + d_\nu (J_r^{[\mu\nu]} c^r) - d_\nu J_r^{[\mu\nu]} c^r] + \\ &\quad (v_{V_r}^A \mathcal{E}_A - d_\nu J_r^{\nu \mu}) c^r \\ &= d_\nu (J_r^{[\mu\nu] \mu_2 \dots \mu_M} c_{\mu_2 \dots \mu_M}^r) + \\ &\quad \sum_{1 < k \leq M-N} [d_\nu (J_r^{[\mu\nu] \mu_{k+1} \dots \mu_M} c_{\mu_{k+1} \dots \mu_M}^r) - d_\nu J_r^{(\nu\mu) \mu_k \dots \mu_M} c_{\mu_k \dots \mu_M}^r] + \\ &\quad \sum_{1 < k < N} [(v_{V_r}^A \mu_{\mu_k \dots \mu_N} \mathcal{E}_A - d_\nu J_r^{(\nu\mu) \mu_k \dots \mu_N}) c_{\mu_k \dots \mu_N}^r + \\ &\quad d_\nu (J_r^{[\mu\nu] \mu_{k+1} \dots \mu_N} c_{\mu_{k+1} \dots \mu_N}^r)] + \\ &\quad [(v_{V_r}^{A \mu\mu_N} \mathcal{E}_A - d_\nu J_r^{(\nu\mu) \mu_N}) c_{\mu_N}^r + d_\nu (J_r^{[\mu\nu]} c^r)] + (v_{V_r}^A \mathcal{E}_A - d_\nu J_r^{(\nu\mu)}) c^r \end{aligned}$$

leads to the expression

$$\begin{aligned} \mathcal{J}_v^\mu = & \left(\sum_{1 < k \leq N} v_{V_r}^{i, \mu \mu_k \dots \mu_N} c_{\mu_k \dots \mu_N}^r + v_{V_r}^{A, \mu} c^r \right) \mathcal{E}_A - \\ & \left(\sum_{1 < k \leq M} d_\nu J^{(\nu \mu) \mu_k \dots \mu_M} c_{\mu_k \dots \mu_M}^r + d_\nu J_r^{(\nu \mu)} c^r \right) - \\ & d_\nu \left(\sum_{1 < k \leq M} J^{[\nu \mu] \mu_k \dots \mu_M} c_{\mu_k \dots \mu_M}^r + J_r^{[\nu \mu]} c^r \right). \end{aligned} \quad (4.42)$$

The first summand of this expression vanishes on-shell. Its second one contains the terms $d_\nu J^{(\nu \mu_k) \mu_{k+1} \dots \mu_M}$, $k = 1, \dots, M$. By virtue of the equalities (4.38) – (4.39), every $d_\nu J^{(\nu \mu_k) \mu_{k+1} \dots \mu_M}$ is expressed into the terms vanishing on-shell and the term $d_\nu J^{(\nu \mu_{k-1}) \mu_k \dots \mu_M}$. Iterating the procedure and bearing in mind the equality (4.37), one can easily show that the second summand of the expression (4.42) also vanishes on-shell. Thus a symmetry current takes the form (4.35) where

$$U^{\nu \mu} = - \sum_{1 < k \leq M} J^{[\nu \mu] \mu_k \dots \mu_M} c_{\mu_k \dots \mu_M}^r - J_r^{[\nu \mu]} c^r. \quad (4.43)$$

□

Example 4.6. If a gauge symmetry

$$v = (v_r^\lambda c^r + v_r^{\lambda \mu} c_\mu^r) \partial_\lambda + (v_r^A c^r + v_r^{A \mu} c_\mu^r) \partial_A \quad (4.44)$$

of a graded Lagrangian L depends at most on the first jets of gauge parameters, then the decomposition (4.42) takes a form

$$\begin{aligned} \mathcal{J}_v^\mu = & v_{V_r}^{A, \mu} \mathcal{E}_A c^r - d_\nu (J_r^{[\nu \mu]} c^r) = \\ & (v_a^{i, \mu} r - s_\lambda^A v_r^{\lambda \mu}) c^r \mathcal{E}_A + d_\nu [(v_r^{A, [\mu} - s_\lambda^A v_r^{\lambda [\mu} c^r \partial_A^{\nu]} \mathcal{L} + v_r^{[\nu \mu]} c^r \mathcal{L}]. \end{aligned} \quad (4.45)$$

5 Second Noether theorems

Let $(\mathcal{S}_\infty^*[F; Y], L)$ be a Grassmann-graded Lagrangian system. Describing Noether and higher-stage identities of its Euler – Lagrange operator, we follow the general analysis of Noether and higher-stage Noether identities of differential operators on fibre bundles in Section 5.5. In the case of an Euler – Lagrange operator as a variation of a Lagrangian, one can however formulate the second Noether theorems (Theorems 5.9 – 5.10) which associate to these identities the gauge and higher-stage gauge symmetries of a Lagrangian system [36, 66].

5.1 Noether and higher-stage Noether identities

Without a lose of generality, let a Lagrangian L be even.

Its Euler – Lagrange operator δL (3.6) is assumed to be at least of order 1 in order to guarantee that transition functions of Y do not vanish on-shell. This Euler – Lagrange operator $\delta L \in \mathbf{E}_1 \subset \mathcal{S}_\infty^{1, n}[F; Y]$ takes its values into the graded vector bundle

$$\overline{VF} = V^* F \otimes_F^n T^* X \rightarrow F, \quad (5.1)$$

where V^*F is the vertical cotangent bundle of $F \rightarrow X$. It however is not a vector bundle over Y . Therefore, we restrict our consideration to the case of the pull-back composite bundle F (2.30) that is

$$F = Y \times_X F^1 \rightarrow Y \rightarrow X, \quad (5.2)$$

where $F^1 \rightarrow X$ is a vector bundle.

Remark 5.1. Let us introduce the following notation. Given the vertical tangent bundle VE of a fibre bundle $E \rightarrow X$, by its density-dual bundle is meant the fibre bundle

$$\overline{VE} = V^*E \otimes_E^n T^*X. \quad (5.3)$$

If $E \rightarrow X$ is a vector bundle, we have

$$\overline{VE} = \overline{E} \times_X E, \quad \overline{E} = E^* \otimes_X^n T^*X, \quad (5.4)$$

where \overline{E} is called the density-dual of E . Let

$$E = E^0 \oplus_X E^1 \quad (5.5)$$

be a graded vector bundle over X . Its graded density-dual is defined to be

$$\overline{E} = \overline{E}^1 \oplus_X \overline{E}^0$$

with an even part $\overline{E}^1 \rightarrow X$ and the odd one $\overline{E}^0 \rightarrow X$. Given the graded vector bundle E (5.5), we consider the product (4.29) of graded bundles over the product (2.52) of the composite bundles E (4.28) and F (2.30) and the corresponding DBGA (4.30) which we denote:

$$P_\infty^*[F \times_X E; Y] = \mathcal{S}_\infty^*[F \times_X E; Y \times_X E^0]. \quad (5.6)$$

In particular, we treat the composite bundle F (2.30) as a graded vector bundle over Y possessing only an odd part. The density-dual (5.3) of the vertical tangent bundle VF of $F \rightarrow X$ is \overline{VF} (5.1). If F is the pull-back bundle (5.2), then

$$\overline{VF} = \overline{F}^1 \oplus_Y ((V^*Y \otimes_Y^n T^*X) \oplus_Y F^1) \quad (5.7)$$

is a graded vector bundle over Y . This bundle can be seen as the product (2.52) of composite bundles

$$\overline{VF^1} = \overline{F}^1 \oplus_X F^1 \rightarrow \overline{F}^1 \rightarrow X, \quad \overline{VY} \rightarrow Y \rightarrow X,$$

and we consider the corresponding graded bundle (2.51) and the DBGA (2.53) which we denote

$$\mathcal{P}_\infty^*[\overline{VF}; Y] = \mathcal{S}_\infty^*[\overline{VF}; Y \times_X \overline{F}^1] = \mathcal{S}_\infty^*[\overline{VF}^1 \times_X \overline{VY}; Y \times_X \overline{F}^1]. \quad (5.8)$$

Lemma 5.1. *One can associate to any Grassmann-graded Lagrangian system $(\mathcal{S}_\infty^*[F; Y], L)$ the chain complex (5.10) whose one-boundaries vanish on-shell.*

Proof. Let us consider the density-dual \overline{VF} (5.7) of the vertical tangent bundle $VF \rightarrow F$, and let us enlarge the original DBGA $\mathcal{S}_\infty^*[F; Y]$ to the DBGA $\mathcal{P}_\infty^*[\overline{VF}; Y]$ (5.8) with the local generating basis

$$(s^A, \bar{s}_A), \quad [\bar{s}_A] = ([A] + 1) \bmod 2.$$

Following the terminology of Lagrangian BRST theory [3, 37], we agree to call its elements \bar{s}_A the Noether antifields of antifield number $\text{Ant}[\bar{s}_A] = 1$. The DBGA $\mathcal{P}_\infty^*[\overline{VF}; Y]$ is endowed with the nilpotent right graded derivation

$$\bar{\delta} = \overleftarrow{\partial}^A \mathcal{E}_A, \quad (5.9)$$

where \mathcal{E}_A are the variational derivatives (3.6). Then we have a chain complex

$$0 \leftarrow \text{Im } \bar{\delta} \xleftarrow{\bar{\delta}} \mathcal{P}_\infty^{0,n}[\overline{VF}; Y]_1 \xleftarrow{\bar{\delta}} \mathcal{P}_\infty^{0,n}[\overline{VF}; Y]_2 \quad (5.10)$$

of graded densities of antifield number ≤ 2 . Its one-boundaries $\bar{\delta}\Phi$, $\Phi \in \mathcal{P}_\infty^{0,n}[\overline{VF}; Y]_2$, by very definition, vanish on-shell. \square

Any one-cycle

$$\Phi = \sum_{0 \leq |\Lambda|} \Phi^{A,\Lambda} \bar{s}_{\Lambda A} \omega \in \mathcal{P}_\infty^{0,n}[\overline{VF}; Y]_1 \quad (5.11)$$

of the complex (5.10) is a differential operator on a bundle \overline{VF} such that it is linear on fibres of $\overline{VF} \rightarrow F$ and its kernel contains the graded Euler – Lagrange operator δL (3.6), i.e.,

$$\bar{\delta}\Phi = 0, \quad \sum_{0 \leq |\Lambda|} \Phi^{A,\Lambda} d_\Lambda \mathcal{E}_A \omega = 0. \quad (5.12)$$

Refereing to Definition 5.16 of Noether identities of a differential operator in Section 5.5, one can say that the one-cycles (5.11) define the Noether identities (5.12) of an Euler – Lagrange operator δL , which we agree to call the Noether identities of a Grassmann-graded Lagrangian system $(\mathcal{S}_\infty^*[F; Y], L)$.

In particular, one-chains Φ (5.11) are necessarily Noether identities if they are boundaries. Therefore, these Noether identities are called trivial. They are of the form

$$\begin{aligned} \Phi &= \sum_{0 \leq |\Lambda|, |\Sigma|} T^{(A\Lambda)(B\Sigma)} d_\Sigma \mathcal{E}_B \bar{s}_{\Lambda A} \omega, \\ T^{(A\Lambda)(B\Sigma)} &= -(-1)^{[A][B]} T^{(B\Sigma)(A\Lambda)}. \end{aligned}$$

Accordingly, non-trivial Noether identities modulo the trivial ones are associated to elements of the first homology $H_1(\bar{\delta})$ of the complex (5.10). A Lagrangian L is called degenerate if there are non-trivial Noether identities.

Non-trivial Noether identities can obey first-stage Noether identities. In order to describe them, let us assume that a module $H_1(\bar{\delta})$ is finitely generated. Namely, there exists a graded projective $C^\infty(X)$ -module $\mathcal{C}_{(0)} \subset H_1(\bar{\delta})$ of finite rank possessing a local basis $\{\Delta_r \omega\}$:

$$\Delta_r \omega = \sum_{0 \leq |\Lambda|} \Delta_r^{A,\Lambda} \bar{s}_{\Lambda A} \omega, \quad \Delta_r^{A,\Lambda} \in \mathcal{S}_\infty^0[F; Y], \quad (5.13)$$

such that any element $\Phi \in H_1(\bar{\delta})$ factorizes as

$$\Phi = \sum_{0 \leq |\Xi|} \Phi^{r,\Xi} d_\Xi \Delta_r \omega, \quad \Phi^{r,\Xi} \in \mathcal{S}_\infty^0[F; Y], \quad (5.14)$$

through elements (5.13) of $\mathcal{C}_{(0)}$. Thus, all non-trivial Noether identities (5.12) result from the Noether identities

$$\bar{\delta}\Delta_r = \sum_{0 \leq |\Lambda|} \Delta_r^{A,\Lambda} d_\Lambda \mathcal{E}_A = 0, \quad (5.15)$$

called the complete Noether identities. Clearly, the factorization (5.14) is independent of specification of a local basis $\{\Delta_r \omega\}$. Note that, being representatives of $H_1(\bar{\delta})$, the graded densities $\Delta_r \omega$ (5.13) are not $\bar{\delta}$ -exact.

A Lagrangian system whose non-trivial Noether identities are finitely generated is called finitely degenerate. Hereafter, degenerate Lagrangian systems only of this type are considered.

Lemma 5.2. *If the homology $H_1(\bar{\delta})$ of the complex (5.10) is finitely generated in the above mentioned sense, this complex can be extended to the one-exact chain complex (5.18) with a boundary operator whose nilpotency conditions are equivalent to the complete Noether identities (5.15).*

Proof. By virtue of Serre – Swan Theorem 2.2, a graded module $\mathcal{C}_{(0)}$ is isomorphic to a module of sections of the density-dual \bar{E}_0 of some graded vector bundle $E_0 \rightarrow X$. Let us enlarge $\mathcal{P}_\infty^*[\bar{V}F; Y]$ to the DBGA

$$\bar{\mathcal{P}}_\infty^* \{0\} = \mathcal{P}_\infty^*[\bar{V}F \times_X \bar{E}_0; Y] = \mathcal{S}_\infty^*[\bar{V}F \times_X \bar{E}_0; \bar{E}_0 \times_X \bar{F}^1 \times_X Y] \quad (5.16)$$

possessing the local generating basis $(s^A, \bar{s}_A, \bar{c}_r)$ where \bar{c}_r are Noether antifields of Grassmann parity

$$[\bar{c}_r] = ([\Delta_r] + 1) \bmod 2$$

and antifield number $\text{Ant}[\bar{c}_r] = 2$. The DBGA (5.16) is provided with the odd right graded derivation

$$\delta_0 = \bar{\delta} + \overleftarrow{\partial}^r \Delta_r \quad (5.17)$$

which is nilpotent iff the complete Noether identities (5.15) hold. Then δ_0 (5.17) is a boundary operator of a chain complex

$$0 \leftarrow \text{Im } \bar{\delta} \xleftarrow{\bar{\delta}} \mathcal{P}_\infty^{0,n}[\bar{V}F; Y]_1 \xleftarrow{\delta_0} \bar{\mathcal{P}}_\infty^{0,n} \{0\}_2 \xleftarrow{\delta_0} \bar{\mathcal{P}}_\infty^{0,n} \{0\}_3 \quad (5.18)$$

of graded densities of antifield number ≤ 3 . Let $H_*(\delta_0)$ denote its homology. We have

$$H_0(\delta_0) = H_0(\bar{\delta}) = 0.$$

Furthermore, any one-cycle Φ up to a boundary takes the form (5.14) and, therefore, it is a δ_0 -boundary

$$\Phi = \sum_{0 \leq |\Sigma|} \Phi^{r,\Xi} d_\Xi \Delta_r \omega = \delta_0 \left(\sum_{0 \leq |\Sigma|} \Phi^{r,\Xi} \bar{c}_{\Xi r} \omega \right).$$

Hence, $H_1(\delta_0) = 0$, i.e., the complex (5.18) is one-exact. \square

Let us consider the second homology $H_2(\delta_0)$ of the complex (5.18). Its two-chains read

$$\Phi = G + H = \sum_{0 \leq |\Lambda|} G^{r,\Lambda} \bar{c}_{\Lambda r} \omega + \sum_{0 \leq |\Lambda|, |\Sigma|} H^{(A,\Lambda)(B,\Sigma)} \bar{s}_{\Lambda A} \bar{s}_{\Sigma B} \omega. \quad (5.19)$$

Its two-cycles define the first-stage Noether identities

$$\delta_0 \Phi = 0, \quad \sum_{0 \leq |\Lambda|} G^{r,\Lambda} d_\Lambda \Delta_r \omega = -\bar{\delta} H. \quad (5.20)$$

Conversely, let the equality (5.20) hold. Then it is a cycle condition of the two-chain (5.19).

Remark 5.2. Note that this definition of first-stage Noether identities is independent on specification of a generating module $\mathcal{C}_{(0)}$. Given a different one, there exists a chain isomorphism between the corresponding complexes (5.18).

The first-stage Noether identities (5.20) are trivial either if a two-cycle Φ (5.19) is a δ_0 -boundary or its summand G vanishes on-shell. Therefore, non-trivial first-stage Noether identities fails to exhaust the second homology $H_2(\delta_0)$ of the complex (5.18) in general.

Lemma 5.3. *Non-trivial first-stage Noether identities modulo the trivial ones are identified with elements of the homology $H_2(\delta_0)$ iff any $\bar{\delta}$ -cycle $\phi \in \bar{\mathcal{P}}_\infty^{0,n}\{0\}_2$ is a δ_0 -boundary.*

Proof. It suffices to show that, if a summand G of a two-cycle Φ (5.19) is $\bar{\delta}$ -exact, then Φ is a boundary. If $G = \bar{\delta}\Psi$, let us write

$$\Phi = \delta_0 \Psi + (\bar{\delta} - \delta_0) \Psi + H. \quad (5.21)$$

Hence, the cycle condition (5.20) reads

$$\delta_0 \Phi = \bar{\delta}((\bar{\delta} - \delta_0) \Psi + H) = 0.$$

Since any $\bar{\delta}$ -cycle $\phi \in \bar{\mathcal{P}}_\infty^{0,n}\{0\}_2$, by assumption, is δ_0 -exact, then

$$(\bar{\delta} - \delta_0) \Psi + H$$

is a δ_0 -boundary. Consequently, Φ (5.21) is δ_0 -exact. Conversely, let $\Phi \in \bar{\mathcal{P}}_\infty^{0,n}\{0\}_2$ be a $\bar{\delta}$ -cycle, i.e.,

$$\bar{\delta} \Phi = 2\Phi^{(A,\Lambda)(B,\Sigma)} \bar{s}_{\Lambda A} \bar{\delta} \bar{s}_{\Sigma B} \omega = 2\Phi^{(A,\Lambda)(B,\Sigma)} \bar{s}_{\Lambda A} d_\Sigma \mathcal{E}_B \omega = 0.$$

It follows that

$$\Phi^{(A,\Lambda)(B,\Sigma)} \bar{\delta} \bar{s}_{\Sigma B} = 0$$

for all indices (A, Λ) . Omitting a $\bar{\delta}$ -boundary term, we obtain

$$\Phi^{(A,\Lambda)(B,\Sigma)} \bar{s}_{\Sigma B} = G^{(A,\Lambda)(r,\Xi)} d_\Xi \Delta_r.$$

Hence, Φ takes a form

$$\Phi = G'^{(A,\Lambda)(r,\Xi)} d_\Xi \Delta_r \bar{s}_{\Lambda A} \omega.$$

Then there exists a three-chain

$$\Psi = G''^{(A,\Lambda)(r,\Xi)} \bar{c}_{\Xi r} \bar{s}_{\Lambda A} \omega$$

such that

$$\delta_0 \Psi = \Phi + \sigma = \Phi + G''^{(A,\Lambda)(r,\Xi)} d_\Lambda \mathcal{E}_A \bar{c}_{\Xi r} \omega. \quad (5.22)$$

Owing to the equality $\bar{\delta}\Phi = 0$, we have $\delta_0\sigma = 0$. Thus, σ in the expression (5.22) is $\bar{\delta}$ -exact δ_0 -cycle. By assumption, it is δ_0 -exact, i.e., $\sigma = \delta_0\psi$. Consequently, a $\bar{\delta}$ -cycle Φ is a δ_0 -boundary

$$\Phi = \delta_0\Psi - \delta_0\psi.$$

□

Remark 5.3. It is easily justified that the two-cycle Φ (5.19) is δ_0 -exact iff Φ up to a $\bar{\delta}$ -boundary takes a form

$$\Phi = \sum_{0 \leq |\Lambda|, |\Sigma|} G'^{(r, \Sigma)(r', \Lambda)} d_\Sigma \Delta_r d_\Lambda \Delta_{r'} \omega.$$

A degenerate Lagrangian system is called reducible if it admits non-trivial first stage Noether identities.

If the condition of Lemma 5.3 is satisfied, let us assume that non-trivial first-stage Noether identities are finitely generated as follows. There exists a graded projective $C^\infty(X)$ -module $\mathcal{C}_{(1)} \subset H_2(\delta_0)$ of finite rank possessing a local basis $\{\Delta_{r_1}\omega\}$:

$$\Delta_{r_1}\omega = \sum_{0 \leq |\Lambda|} \Delta_{r_1}^{r, \Lambda} \bar{c}_{\Lambda r} \omega + h_{r_1} \omega, \quad (5.23)$$

such that any element $\Phi \in H_2(\delta_0)$ factorizes as

$$\Phi = \sum_{0 \leq |\Xi|} \Phi^{r_1, \Xi} d_\Xi \Delta_{r_1} \omega, \quad \Phi^{r_1, \Xi} \in \mathcal{S}_\infty^0[F; Y], \quad (5.24)$$

through elements (5.23) of $\mathcal{C}_{(1)}$. Thus, all non-trivial first-stage Noether identities (5.20) result from the equalities

$$\sum_{0 \leq |\Lambda|} \Delta_{r_1}^{r, \Lambda} d_\Lambda \Delta_r + \bar{\delta} h_{r_1} = 0, \quad (5.25)$$

called the complete first-stage Noether identities. Note that, by virtue of the condition of Lemma 5.3, the first summands of the graded densities $\Delta_{r_1}\omega$ (5.23) are not $\bar{\delta}$ -exact.

A degenerate Lagrangian system is called finitely reducible if it admits finitely generated non-trivial first-stage Noether identities.

Lemma 5.4. *The one-exact complex (5.18) of a finitely reducible Lagrangian system is extended to the two-exact one (5.27) with a boundary operator whose nilpotency conditions are equivalent to the complete Noether identities (5.15) and the complete first-stage Noether identities (5.25).*

Proof. By virtue of Serre – Swan Theorem 2.2, a graded module $\mathcal{C}_{(1)}$ is isomorphic to a module of sections of the density-dual \bar{E}_1 of some graded vector bundle $E_1 \rightarrow X$. Let us enlarge the DBGA $\bar{\mathcal{P}}_\infty^*\{0\}$ (5.16) to the DBGA

$$\bar{\mathcal{P}}_\infty^*\{1\} = \mathcal{P}_\infty^*[\bar{V}F \times_X \bar{E}_0 \times_X \bar{E}_1; Y]$$

possessing the local generating basis $\{s^A, \bar{s}_A, \bar{c}_r, \bar{c}_{r_1}\}$ where \bar{c}_{r_1} are first stage Noether antifields of Grassmann parity

$$[\bar{c}_{r_1}] = ([\Delta_{r_1}] + 1) \bmod 2$$

and antifield number $\text{Ant}[\bar{c}_{r_1}] = 3$. This DBGA is provided with the odd right graded derivation

$$\delta_1 = \delta_0 + \overset{\leftarrow}{\partial}{}^{r_1} \Delta_{r_1} \quad (5.26)$$

which is nilpotent iff the complete Noether identities (5.15) and the complete first-stage Noether identities (5.97) hold. Then δ_1 (5.26) is a boundary operator of the chain complex

$$0 \leftarrow \text{Im } \bar{\delta} \xleftarrow{\bar{\delta}} \mathcal{P}_\infty^{0,n}[\overline{VF}; Y]_1 \xleftarrow{\delta_0} \overline{\mathcal{P}}_\infty^{0,n}\{0\}_2 \xleftarrow{\delta_1} \overline{\mathcal{P}}_\infty^{0,n}\{1\}_3 \xleftarrow{\delta_1} \overline{\mathcal{P}}_\infty^{0,n}\{1\}_4 \quad (5.27)$$

of graded densities of antifield number ≤ 4 . Let $H_*(\delta_1)$ denote its homology. It is readily observed that

$$H_0(\delta_1) = H_0(\bar{\delta}), \quad H_1(\delta_1) = H_1(\delta_0) = 0.$$

By virtue of the expression (5.24), any two-cycle of the complex (5.27) is a boundary

$$\Phi = \sum_{0 \leq |\Xi|} \Phi^{r_1, \Xi} d_\Xi \Delta_{r_1} \omega = \delta_1 \left(\sum_{0 \leq |\Xi|} \Phi^{r_1, \Xi} \bar{c}_{\Xi r_1} \omega \right).$$

It follows that $H_2(\delta_1) = 0$, i.e., the complex (5.27) is two-exact. \square

If the third homology $H_3(\delta_1)$ of the complex (5.27) is not trivial, its elements correspond to second-stage Noether identities which the complete first-stage ones satisfy, and so on. Iterating the arguments, one comes to the following.

A degenerate Grassmann-graded Lagrangian system $(\mathcal{S}_\infty^*[F; Y], L)$ is called N -stage reducible if it admits finitely generated non-trivial N -stage Noether identities, but no non-trivial $(N+1)$ -stage ones. It is characterized as follows [8, 36].

- There are graded vector bundles E_0, \dots, E_N over X , and a DBGA $\mathcal{P}_\infty^*[\overline{VF}; Y]$ is enlarged to a DBGA

$$\overline{\mathcal{P}}_\infty^*\{N\} = \mathcal{P}_\infty^*[\overline{VF} \times_X \overline{E}_0 \times_X \dots \times_X \overline{E}_N; Y] \quad (5.28)$$

with the local generating basis

$$(s^A, \bar{s}_A, \bar{c}_r, \bar{c}_{r_1}, \dots, \bar{c}_{r_N})$$

where \bar{c}_{r_k} are k -stage Noether antifields of antifield number $\text{Ant}[\bar{c}_{r_k}] = k + 2$.

- The DBGA (5.28) is provided with a nilpotent right graded derivation

$$\delta_{\text{KT}} = \delta_N = \bar{\delta} + \sum_{0 \leq |\Lambda|} \overset{\leftarrow}{\partial}{}^r \Delta_r^{A, \Lambda} \bar{s}_{\Lambda A} + \sum_{1 \leq k \leq N} \overset{\leftarrow}{\partial}{}^{r_k} \Delta_{r_k}, \quad (5.29)$$

$$\Delta_{r_k} \omega = \sum_{0 \leq |\Lambda|} \Delta_{r_k}^{r_{k-1}, \Lambda} \bar{c}_{\Lambda r_{k-1}} \omega + \quad (5.30)$$

$$\sum_{0 \leq |\Sigma|, |\Xi|} (h_{r_k}^{(r_{k-2}, \Sigma)(A, \Xi)} \bar{c}_{\Sigma r_{k-2}} \bar{s}_{\Xi A} + \dots) \omega \in \overline{\mathcal{P}}_\infty^{0,n}\{k-1\}_{k+1},$$

of antifield number -1. The index $k = -1$ here stands for \bar{s}_A . The nilpotent derivation δ_{KT} (5.29) is called the Koszul – Tate operator.

- With this graded derivation, a module $\overline{\mathcal{P}}_\infty^{0,n}\{N\}_{\leq N+3}$ of densities of antifield number $\leq (N+3)$ is decomposed into the exact Koszul – Tate chain complex

$$0 \leftarrow \text{Im } \bar{\delta} \xleftarrow{\bar{\delta}} \mathcal{P}_\infty^{0,n}[\overline{VF}; Y]_1 \xleftarrow{\delta_0} \overline{\mathcal{P}}_\infty^{0,n}\{0\}_2 \xleftarrow{\delta_1} \overline{\mathcal{P}}_\infty^{0,n}\{1\}_3 \dots \quad (5.31)$$

$$\xleftarrow{\delta_{N-1}} \overline{\mathcal{P}}_\infty^{0,n}\{N-1\}_{N+1} \xleftarrow{\delta_{\text{KT}}} \overline{\mathcal{P}}_\infty^{0,n}\{N\}_{N+2} \xleftarrow{\delta_{\text{KT}}} \overline{\mathcal{P}}_\infty^{0,n}\{N\}_{N+3}$$

which satisfies the following homology regularity condition.

Condition 5.5. Any $\delta_{k < N}$ -cycle

$$\phi \in \overline{\mathcal{P}}_{\infty}^{0,n} \{k\}_{k+3} \subset \overline{\mathcal{P}}_{\infty}^{0,n} \{k+1\}_{k+3}$$

is a δ_{k+1} -boundary.

Remark 5.4. The exactness of the complex (5.31) means that any $\delta_{k < N}$ -cycle $\phi \in \mathcal{P}_{\infty}^{0,n} \{k\}_{k+3}$, is a δ_{k+2} -boundary, but not necessary a δ_{k+1} -one.

• The nilpotentness $\delta_{\text{KT}}^2 = 0$ of the Koszul – Tate operator (5.29) is equivalent to the complete non-trivial Noether identities (5.15) and the complete non-trivial ($k \leq N$)-stage Noether identities

$$\begin{aligned} \sum_{0 \leq |\Lambda|} \Delta_{r_k}^{r_{k-1}, \Lambda} d_{\Lambda} \left(\sum_{0 \leq |\Sigma|} \Delta_{r_{k-1}}^{r_{k-2}, \Sigma} \overline{c}_{\Sigma r_{k-2}} \right) = \\ -\overline{\delta} \left(\sum_{0 \leq |\Sigma|, |\Xi|} h_{r_k}^{(r_{k-2}, \Sigma)(A, \Xi)} \overline{c}_{\Sigma r_{k-2}} \overline{s}_{\Xi A} \right). \end{aligned} \quad (5.32)$$

This item means the following.

Lemma 5.6. Any δ_k -cocycle $\Phi \in \mathcal{P}_{\infty}^{0,n} \{k\}_{k+2}$ is a k -stage Noether identity, and vice versa.

Proof. Any $(k+2)$ -chain $\Phi \in \mathcal{P}_{\infty}^{0,n} \{k\}_{k+2}$ takes a form

$$\begin{aligned} \Phi = G + H = \sum_{0 \leq |\Lambda|} G^{r_k, \Lambda} \overline{c}_{\Lambda r_k} \omega + \\ \sum_{0 \leq \Sigma, 0 \leq \Xi} (H^{(A, \Xi)(r_{k-1}, \Sigma)} \overline{s}_{\Xi A} \overline{c}_{\Sigma r_{k-1}} + \dots) \omega. \end{aligned} \quad (5.33)$$

If it is a δ_k -cycle, then

$$\begin{aligned} \sum_{0 \leq |\Lambda|} G^{r_k, \Lambda} d_{\Lambda} \left(\sum_{0 \leq |\Sigma|} \Delta_{r_k}^{r_{k-1}, \Sigma} \overline{c}_{\Sigma r_{k-1}} \right) + \\ \overline{\delta} \left(\sum_{0 \leq \Sigma, 0 \leq \Xi} H^{(A, \Xi)(r_{k-1}, \Sigma)} \overline{s}_{\Xi A} \overline{c}_{\Sigma r_{k-1}} \right) = 0 \end{aligned} \quad (5.34)$$

are the k -stage Noether identities. Conversely, let the condition (5.34) hold. Then it can be extended to a cycle condition as follows. It is brought into the form

$$\begin{aligned} \delta_k \left(\sum_{0 \leq |\Lambda|} G^{r_k, \Lambda} \overline{c}_{\Lambda r_k} + \sum_{0 \leq \Sigma, 0 \leq \Xi} H^{(A, \Xi)(r_{k-1}, \Sigma)} \overline{s}_{\Xi A} \overline{c}_{\Sigma r_{k-1}} \right) = \\ - \sum_{0 \leq |\Lambda|} G^{r_k, \Lambda} d_{\Lambda} h_{r_k} + \sum_{0 \leq \Sigma, 0 \leq \Xi} H^{(A, \Xi)(r_{k-1}, \Sigma)} \overline{s}_{\Xi A} d_{\Sigma} \Delta_{r_{k-1}}. \end{aligned}$$

A glance at the expression (5.30) shows that the term in the right-hand side of this equality belongs to $\mathcal{P}_{\infty}^{0,n} \{k-2\}_{k+1}$. It is a δ_{k-2} -cycle and, consequently, a δ_{k-1} -boundary $\delta_{k-1} \Psi$ in

accordance with Condition 5.5. Then the equality (5.34) is a $\bar{\tau}_{\Sigma r_{k-1}}$ -dependent part of the cycle condition

$$\delta_k \left(\sum_{0 \leq |\Lambda|} G^{r_k, \Lambda} \bar{c}_{\Lambda r_k} + \sum_{0 \leq \Sigma, 0 \leq \Xi} H^{(A, \Xi)(r_{k-1}, \Sigma)} \bar{s}_{\Xi A} \bar{c}_{\Sigma r_{k-1}} - \Psi \right) = 0,$$

but $\delta_k \Psi$ does not make a contribution to this condition. \square

Lemma 5.7. *Any trivial k -stage Noether identity is a δ_k -boundary $\Phi \in \mathcal{P}_{\infty}^{0,n}\{k\}_{k+2}$.*

Proof. The k -stage Noether identities (5.34) are trivial either if a δ_k -cycle Φ (5.33) is a δ_k -boundary or its summand G vanishes on-shell. Let us show that, if the summand G of Φ (5.33) is $\bar{\delta}$ -exact, then Φ is a δ_k -boundary. If $G = \bar{\delta}\Psi$, one can write

$$\Phi = \delta_k \Psi + (\bar{\delta} - \delta_k) \Psi + H. \quad (5.35)$$

Hence, the δ_k -cycle condition reads

$$\delta_k \Phi = \delta_{k-1}((\bar{\delta} - \delta_k) \Psi + H) = 0.$$

By virtue of Condition 5.5, any δ_{k-1} -cycle $\phi \in \bar{\mathcal{P}}_{\infty}^{0,n}\{k-1\}_{k+2}$ is δ_k -exact. Then

$$(\bar{\delta} - \delta_k) \Psi + H$$

is a δ_k -boundary. Consequently, Φ (5.33) is δ_k -exact. \square

Lemma 5.8. *All non-trivial k -stage Noether identity (5.34), by assumption, factorize as*

$$\Phi = \sum_{0 \leq |\Xi|} \Phi^{r_k, \Xi} d_{\Xi} \Delta_{r_k} \omega, \quad \Phi^{r_1, \Xi} \in \mathcal{S}_{\infty}^0[F; Y],$$

through the complete ones (5.32).

It may happen that a Grassmann-graded Lagrangian field system possesses non-trivial Noether identities of any stage. However, we restrict our consideration to N -reducible Lagrangian systems for a finite integer N . In this case, the Koszul – Tate operator (5.29) and the gauge operator (5.43) below contain finite terms.

5.2 Inverse second Noether theorem

Different variants of the second Noether theorem have been suggested in order to relate reducible Noether identities and gauge symmetries [3, 6, 7, 27]. The inverse second Noether theorem (Theorem 5.9), that we formulate in homology terms, associates to the Koszul – Tate complex (5.31) of non-trivial Noether identities the cochain sequence (5.42) with the ascent operator \mathbf{u} (5.43) whose components are complete non-trivial gauge and higher-stage gauge symmetries of Lagrangian system. Let us start with the following notation.

Remark 5.5. Given the DBGA $\bar{\mathcal{P}}_{\infty}^*\{N\}$ (5.28), we consider the the DBGA

$$P_{\infty}^*\{N\} = P_{\infty}^*[F \times_X E_0 \times_X \cdots \times_X E_N; Y], \quad (5.36)$$

possessing the local generating basis

$$(s^A, c^r, c^{r_1}, \dots, c^{r_N}), \quad [c^{r_k}] = ([\bar{c}_{r_k}] + 1) \bmod 2,$$

and the DBGA

$$\mathcal{P}_\infty^*\{N\} = \mathcal{P}_\infty^*[\overline{VF} \times_X E_0 \times_X \dots \times_X E_N \times_X \overline{E}_0 \times_X \dots \times_X \overline{E}_N; Y] \quad (5.37)$$

with the local generating basis

$$(s^A, \bar{s}_A, c^r, c^{r_1}, \dots, c^{r_N}, \bar{c}_r, \bar{c}_{r_1}, \dots, \bar{c}_{r_N}),$$

(see Remark 5.1 for the notation). Their elements c^{r_k} are called k -stage ghosts of ghost number $\text{gh}[c^{r_k}] = k + 1$ and antifield number

$$\text{Ant}[c^{r_k}] = -(k + 1).$$

A $C^\infty(X)$ -module $\mathcal{C}^{(k)}$ of k -stage ghosts is the density-dual of a module $\mathcal{C}_{(k+1)}$ of $(k + 1)$ -stage Noether antifields. In accordance with Remark 2.6, the DBGAs $\overline{\mathcal{P}}_\infty^*\{N\}$ (5.28) and the BGDA $P^*(N)$ (5.36) are subalgebras of the DBGA $\mathcal{P}_\infty^*\{N\}$ (5.37). The Koszul – Tate operator δ_{KT} (5.29) is naturally extended to a graded derivation of the DBGA $\mathcal{P}_\infty^*\{N\}$.

Remark 5.6. Any graded differential form $\phi \in \mathcal{S}_\infty^*[F; Y]$ and any finite tuple (f^Λ) , $0 \leq |\Lambda| \leq k$, of local graded functions $f^\Lambda \in \mathcal{S}_\infty^0[F; Y]$ obey the following relations [7]:

$$\sum_{0 \leq |\Lambda| \leq k} f^\Lambda d_\Lambda \phi \wedge \omega = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} d_\Lambda (f^\Lambda) \phi \wedge \omega + d_H \sigma, \quad (5.38)$$

$$\sum_{0 \leq |\Lambda| \leq k} (-1)^{|\Lambda|} d_\Lambda (f^\Lambda \phi) = \sum_{0 \leq |\Lambda| \leq k} \eta(f)^\Lambda d_\Lambda \phi, \quad (5.39)$$

$$\eta(f)^\Lambda = \sum_{0 \leq |\Sigma| \leq k - |\Lambda|} (-1)^{|\Sigma + \Lambda|} \frac{(|\Sigma + \Lambda|)!}{|\Sigma|! |\Lambda|!} d_\Sigma f^{\Sigma + \Lambda}, \quad (5.40)$$

$$\eta(\eta(f))^\Lambda = f^\Lambda. \quad (5.41)$$

Theorem 5.9. *Given the Koszul – Tate complex (5.31), a module of graded densities $P_\infty^{0,n}\{N\}$ is decomposed into the cochain sequence*

$$0 \rightarrow \mathcal{S}_\infty^{0,n}[F; Y] \xrightarrow{\mathbf{u}} P_\infty^{0,n}\{N\}^1 \xrightarrow{\mathbf{u}} P_\infty^{0,n}\{N\}^2 \xrightarrow{\mathbf{u}} \dots, \quad (5.42)$$

$$\mathbf{u} = u + u^{(1)} + \dots + u^{(N)} = \quad (5.43)$$

$$u^A \frac{\partial}{\partial s^A} + u^r \frac{\partial}{\partial c^r} + \dots + u^{r_{N-1}} \frac{\partial}{\partial c^{r_{N-1}}},$$

graded in ghost number. Its ascent operator \mathbf{u} (5.43) is an odd graded derivation of ghost number 1 where u (5.48) is a variational symmetry of a graded Lagrangian L and the graded derivations $u_{(k)}$ (5.51), $k = 1, \dots, N$, obey the relations (5.50).

Proof. Given the Koszul – Tate operator (5.29), let us extend an original Lagrangian L to the Lagrangian

$$L_e = L + L_1 = L + \sum_{0 \leq k \leq N} c^{r_k} \Delta_{r_k} \omega = L + \delta_{\text{KT}} \left(\sum_{0 \leq k \leq N} c^{r_k} \bar{c}_{r_k} \omega \right) \quad (5.44)$$

of zero antifield number. It is readily observed that the Koszul – Tate operator δ_{KT} is an exact symmetry of the extended Lagrangian $L_e \in \mathcal{P}_\infty^{0,n}\{N\}$ (5.44). Since the graded derivation δ_{KT} is vertical, it follows from the first variational formula (4.15) that

$$\begin{aligned}
& \left[\frac{\overleftarrow{\delta} \mathcal{L}_e}{\delta \bar{s}_A} \mathcal{E}_A + \sum_{0 \leq k \leq N} \frac{\overleftarrow{\delta} \mathcal{L}_e}{\delta \bar{c}_{r_k}} \Delta_{r_k} \right] \omega = \\
& \left[v^A \mathcal{E}_A + \sum_{0 \leq k \leq N} v^{r_k} \frac{\delta \mathcal{L}_e}{\delta c^{r_k}} \right] \omega = d_H \sigma, \\
v^A &= \frac{\overleftarrow{\delta} \mathcal{L}_e}{\delta \bar{s}_A} = u^A + w^A = \sum_{0 \leq |\Lambda|} c_\Lambda^r \eta(\Delta_r^A)^\Lambda + \sum_{1 \leq i \leq N} \sum_{0 \leq |\Lambda|} c_\Lambda^{r_i} \eta(\overleftarrow{\partial}^A(h_{r_i}))^\Lambda, \\
v^{r_k} &= \frac{\overleftarrow{\delta} \mathcal{L}_e}{\delta \bar{c}_{r_k}} = u^{r_k} + w^{r_k} = \sum_{0 \leq |\Lambda|} c_\Lambda^{r_{k+1}} \eta(\Delta_{r_{k+1}}^{r_k})^\Lambda + \\
& \sum_{k+1 \leq i \leq N} \sum_{0 \leq |\Lambda|} c_\Lambda^{r_i} \eta(\overleftarrow{\partial}^{r_k}(h_{r_i}))^\Lambda.
\end{aligned} \tag{5.45}$$

The equality (5.45) is split into the set of equalities

$$\frac{\overleftarrow{\delta}(c^r \Delta_r)}{\delta \bar{s}_A} \mathcal{E}_A \omega = u^A \mathcal{E}_A \omega = d_H \sigma_0, \tag{5.46}$$

$$\left[\frac{\overleftarrow{\delta}(c^{r_k} \Delta_{r_k})}{\delta \bar{s}_A} \mathcal{E}_A + \sum_{0 \leq i < k} \frac{\overleftarrow{\delta}(c^{r_k} \Delta_{r_k})}{\delta \bar{c}_{r_i}} \Delta_{r_i} \right] \omega = d_H \sigma_k, \tag{5.47}$$

where $k = 1, \dots, N$. A glance at the equality (5.46) shows that, by virtue of the first variational formula (4.15), the odd graded derivation

$$u = u^A \frac{\partial}{\partial s^A}, \quad u^A = \sum_{0 \leq |\Lambda|} c_\Lambda^r \eta(\Delta_r^A)^\Lambda, \tag{5.48}$$

of $P_\infty^0\{0\}$ is a variational symmetry of a graded Lagrangian L . Every equality (5.47) falls into a set of equalities graded by the polynomial degree in antifields. Let us consider that of them linear in antifields $\bar{c}_{r_{k-2}}$. We have

$$\begin{aligned}
& \frac{\overleftarrow{\delta}}{\delta \bar{s}_A} \left(c^{r_k} \sum_{0 \leq |\Sigma|, |\Xi|} h_{r_k}^{(r_{k-2}, \Sigma)(A, \Xi)} \bar{c}_{\Sigma r_{k-2}} \bar{s}_{\Xi A} \right) \mathcal{E}_A \omega + \\
& \frac{\overleftarrow{\delta}}{\delta \bar{c}_{r_{k-1}}} \left(c^{r_k} \sum_{0 \leq |\Sigma|} \Delta_{r_k}^{r'_{k-1}, \Sigma} \bar{c}_{\Sigma r'_{k-1}} \right) \sum_{0 \leq |\Xi|} \Delta_{r_{k-1}}^{r_{k-2}, \Xi} \bar{c}_{\Xi r_{k-2}} \omega = d_H \sigma_k.
\end{aligned}$$

This equality is brought into the form

$$\begin{aligned}
& \sum_{0 \leq |\Xi|} (-1)^{|\Xi|} d_\Xi \left(c^{r_k} \sum_{0 \leq |\Sigma|} h_{r_k}^{(r_{k-2}, \Sigma)(A, \Xi)} \bar{c}_{\Sigma r_{k-2}} \right) \mathcal{E}_A \omega + \\
& u^{r_{k-1}} \sum_{0 \leq |\Xi|} \Delta_{r_{k-1}}^{r_{k-2}, \Xi} \bar{c}_{\Xi r_{k-2}} \omega = d_H \sigma_k.
\end{aligned}$$

Using the relation (5.38), we obtain the equality

$$\begin{aligned} \sum_{0 \leq |\Xi|} c^{r_k} \sum_{0 \leq |\Sigma|} h_{r_k}^{(r_{k-2}, \Sigma)(A, \Xi)} \bar{c}_{\Sigma r_{k-2}} d_{\Xi} \mathcal{E}_A \omega + \\ u^{r_{k-1}} \sum_{0 \leq |\Xi|} \Delta_{r_{k-1}}^{r_{k-2}, \Xi} \bar{c}_{\Xi r_{k-2}} \omega = d_H \sigma'_k. \end{aligned} \quad (5.49)$$

The variational derivative of both its sides with respect to $\bar{c}_{r_{k-2}}$ leads to the relation

$$\begin{aligned} \sum_{0 \leq |\Sigma|} d_{\Sigma} u^{r_{k-1}} \frac{\partial}{\partial c_{\Sigma}^{r_{k-1}}} u^{r_{k-2}} = \bar{\delta}(\alpha^{r_{k-2}}), \\ \alpha^{r_{k-2}} = - \sum_{0 \leq |\Sigma|} \eta(h_{r_k}^{(r_{k-2})(A, \Xi)})^{\Sigma} d_{\Sigma}(c^{r_k} \bar{s}_{\Xi A}), \end{aligned} \quad (5.50)$$

which the odd graded derivation

$$u^{(k)} = u^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}} = \sum_{0 \leq |\Lambda|} c_{\Lambda}^{r_k} \eta(\Delta_{r_k}^{r_{k-1}})^{\Lambda} \frac{\partial}{\partial c^{r_{k-1}}}, \quad k = 1, \dots, N, \quad (5.51)$$

satisfies. Graded derivations u (5.48) and $u^{(k)}$ (5.51) are assembled into the ascent operator \mathbf{u} (5.43) of the cochain sequence (5.42). \square

A glance at the expression (5.48) shows that the variational symmetry u is a graded derivation of a ring

$$P_{\infty}^0[0] = S_{\infty}^0[F \times_X E_0; Y \times_X E_0^0]$$

which satisfies Definition 4.8. Consequently, u (5.48) is a gauge symmetry of a graded Lagrangian L which is associated to the complete non-trivial Noether identities (5.15). Therefore, it is a non-trivial gauge symmetry. Moreover, it is complete in the following sense. Let

$$\sum_{0 \leq |\Xi|} C^R G_R^{r, \Xi} d_{\Xi} \Delta_r \omega$$

be some projective $C^{\infty}(X)$ -module of finite rank of non-trivial Noether identities (5.14) parameterized by the corresponding ghosts C^R . We have the equalities

$$\begin{aligned} 0 &= \sum_{0 \leq |\Xi|} C^R G_R^{r, \Xi} d_{\Xi} \left(\sum_{0 \leq |\Lambda|} \Delta_r^{A, \Lambda} d_{\Lambda} \mathcal{E}_A \right) \omega = \\ &= \sum_{0 \leq |\Lambda|} \left(\sum_{0 \leq |\Xi|} \eta(G_R^r)^{\Xi} C_{\Xi}^R \right) \Delta_r^{A, \Lambda} d_{\Lambda} \mathcal{E}_A \omega + d_H(\sigma) = \\ &= \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} d_{\Lambda} \left(\Delta_r^{A, \Lambda} \sum_{0 \leq |\Xi|} \eta(G_R^r)^{\Xi} C_{\Xi}^R \right) \mathcal{E}_A \omega + d_H \sigma = \\ &= \sum_{0 \leq |\Lambda|} \eta(\Delta_r^A)^{\Lambda} d_{\Lambda} \left(\sum_{0 \leq |\Xi|} \eta(G_R^r)^{\Xi} C_{\Xi}^R \right) \mathcal{E}_A \omega + d_H \sigma = \\ &= \sum_{0 \leq |\Lambda|} u_r^{A, \Lambda} d_{\Lambda} \left(\sum_{0 \leq |\Xi|} \eta(G_R^r)^{\Xi} C_{\Xi}^R \right) \mathcal{E}_A \omega + d_H \sigma. \end{aligned}$$

It follows that the graded derivation

$$d_\Lambda \left(\sum_{0 \leq |\Xi|} \eta(G_R^r)^\Xi C_\Xi^R \right) u_r^{A,\Lambda} \frac{\partial}{\partial s^A}$$

is a variational symmetry of a graded Lagrangian L and, consequently, its gauge symmetry parameterized by ghosts C^R . It factorizes through the gauge symmetry (5.48) by putting ghosts

$$c^r = \sum_{0 \leq |\Xi|} \eta(G_R^r)^\Xi C_\Xi^R.$$

Thus, we come to the following definition.

Turn now to the relation (5.50). For $k = 1$, it takes a form

$$\sum_{0 \leq |\Sigma|} d_\Sigma u^r \frac{\partial}{\partial c_\Sigma^r} u^A = \bar{\delta}(\alpha^A)$$

of a first-stage gauge symmetry condition on-shell which the non-trivial gauge symmetry u (5.48) satisfies. Therefore, one can treat the odd graded derivation

$$u^{(1)} = u^r \frac{\partial}{\partial c^r}, \quad u^r = \sum_{0 \leq |\Lambda|} c_\Lambda^{r_1} \eta(\Delta_{r_1}^r)^\Lambda,$$

as a first-stage gauge symmetry associated to the complete first-stage Noether identities

$$\sum_{0 \leq |\Lambda|} \Delta_{r_1}^{r,\Lambda} d_\Lambda \left(\sum_{0 \leq |\Sigma|} \Delta_r^{A,\Sigma} \bar{s}_{\Sigma A} \right) = -\bar{\delta} \left(\sum_{0 \leq |\Sigma|, |\Xi|} h_{r_1}^{(B,\Sigma)(A,\Xi)} \bar{s}_{\Sigma B} \bar{s}_{\Xi A} \right).$$

Iterating the arguments, one comes to the relation (5.50) which provides a k -stage gauge symmetry condition which is associated to the complete non-trivial k -stage Noether identities (5.32).

The odd graded derivation $u_{(k)}$ (5.51) is called the k -stage gauge symmetry. It is non-trivial and complete as follows. Let

$$\sum_{0 \leq |\Xi|} C^{R_k} G_{R_k}^{r_k, \Xi} d_\Xi \Delta_{r_k} \omega$$

be a projective $C^\infty(X)$ -module of finite rank of non-trivial k -stage Noether identities (5.14) factorizing through the complete ones (5.30) and parameterized by the corresponding ghosts C^{R_k} . One can show that it defines a k -stage gauge symmetry factorizing through $u^{(k)}$ (5.51) by putting k -stage ghosts

$$c^{r_k} = \sum_{0 \leq |\Xi|} \eta(G_{R_k}^{r_k})^\Xi C_\Xi^{R_k}.$$

Thus, components of the ascent operator \mathbf{u} (5.43) in inverse second Noether Theorem 5.9 are complete non-trivial gauge and higher-stage gauge symmetries. Therefore, we agree to call this operator the gauge operator.

Remark 5.7. With the gauge operator (5.43), the extended Lagrangian L_e (5.44) takes a form

$$L_e = L + \mathbf{u} \left(\sum_{0 \leq k \leq N} c^{r_{k-1}} \bar{c}_{r_{k-1}} \right) \omega + L_1^* + d_H \sigma, \quad (5.52)$$

where L_1^* is a term of polynomial degree in antifields exceeding 1.

5.3 Direct second Noether theorem

The correspondence between of complete non-trivial gauge and higher-stage gauge symmetries to complete non-trivial Noether and higher-stage Noether identities is unique due to the following direct second Noether theorem.

Theorem 5.10. (i) If u (5.48) is a gauge symmetry, the variational derivative of the d_H -exact density $u^A \mathcal{E}_A \omega$ (5.46) with respect to ghosts c^r leads to the equality

$$\begin{aligned} \delta_r(u^A \mathcal{E}_A \omega) &= \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} d_\Lambda [u_r^{A\Lambda} \mathcal{E}_A] = \\ &= \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} d_\Lambda (\eta(\Delta_r^A)^\Lambda \mathcal{E}_A) = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} \eta(\eta(\Delta_r^A))^\Lambda d_\Lambda \mathcal{E}_A = 0, \end{aligned} \quad (5.53)$$

which reproduces the complete Noether identities (5.15) by means of the relation (5.41).

(ii) Given the k -stage gauge symmetry condition (5.50), the variational derivative of the equality (5.49) with respect to ghosts c^{r^k} leads to the equality, reproducing the k -stage Noether identities (5.32) by means of the relations (5.39) – (5.41).

Example 5.8. If the gauge symmetry u (4.34) is of second jet order in gauge parameters, i.e.,

$$u_V = (u_r^A c^r + u_r^{A\mu} c_\mu^r + u_r^{A\nu\mu} c_{\nu\mu}^r) \partial_A, \quad (5.54)$$

the corresponding Noether identities (5.53) take a form

$$u_r^A \mathcal{E}_A - d_\mu (u_r^{A\mu} \mathcal{E}_A) + d_{\nu\mu} (u_r^{A\nu\mu} \mathcal{E}_A) = 0, \quad (5.55)$$

and *vice versa*.

Remark 5.9. A glance at the expression (5.55) shows that, if the gauge symmetry u_V (5.54) is independent of jets of gauge parameters, then all variational derivatives of a Lagrangian equals zero, i.e., this Lagrangian is variationally trivial. Therefore, such gauge symmetries usually are not considered. At the same time, let a Lagrangian L be variationally trivial. Its variational derivatives $\mathcal{E}_A \equiv 0$ obey irreducible complete Noether identities

$$\bar{\delta} \Delta_A = 0, \quad \Delta_A = \bar{s}_A. \quad (5.56)$$

By the formula (5.43), the associated irreducible gauge symmetry is given by the gauge operator

$$\mathbf{u} = c^A \frac{\partial}{\partial s^A}. \quad (5.57)$$

Remark 5.10. One can consider gauge symmetries which need not be linear in gauge parameters. Let us call then the generalized gauge symmetries. However, direct second Noether Theorem 5.53 is not relevant to generalized gauge symmetries because, in this case, an Euler – Lagrange operator satisfies the identities depending on gauge parameters.

5.4 BRST operator

In contrast with the Koszul – Tate operator (5.29), the gauge operator \mathbf{u} (5.42) need not be nilpotent. Following the basic example of Yang – Mills gauge theory (Section 6.1), let us study its extension to a nilpotent graded derivation

$$\begin{aligned} \mathbf{b} &= \mathbf{u} + \gamma = \mathbf{u} + \sum_{1 \leq k \leq N+1} \gamma^{(k)} = \mathbf{u} + \sum_{1 \leq k \leq N+1} \gamma^{r_{k-1}} \frac{\partial}{\partial c^{r_{k-1}}} \\ &= \left(u^A \frac{\partial}{\partial s^A} + \gamma^r \frac{\partial}{\partial c^r} \right) + \sum_{0 \leq k \leq N-1} \left(u^{r_k} \frac{\partial}{\partial c^{r_k}} + \gamma^{r_{k+1}} \frac{\partial}{\partial c^{r_{k+1}}} \right) \end{aligned} \quad (5.58)$$

of ghost number 1 by means of antifield-free terms $\gamma^{(k)}$ of higher polynomial degree in ghosts c^{r_i} and their jets $c_\Lambda^{r_i}$, $0 \leq i < k$. We call \mathbf{b} (5.58) the BRST operator, where k -stage gauge symmetries are extended to k -stage BRST transformations acting both on $(k-1)$ -stage and k -stage ghosts [35, 36]. If the BRST operator exists, the cochain sequence (5.42) is brought into the BRST complex

$$0 \rightarrow \mathcal{S}_\infty^{0,n}[F; Y] \xrightarrow{\mathbf{b}} P_\infty^{0,n}\{N\}^1 \xrightarrow{\mathbf{b}} P_\infty^{0,n}\{N\}^2 \xrightarrow{\mathbf{b}} \dots$$

There is the following necessary condition of the existence of such a BRST extension.

Theorem 5.11. *The gauge operator (5.42) admits the nilpotent extension (5.58) only if the gauge symmetry conditions (5.50) and the higher-stage Noether identities (5.32) are satisfied off-shell.*

Proof. It is easily justified that, if the graded derivation \mathbf{b} (5.58) is nilpotent, then the right hand sides of the equalities (5.50) equal zero, i.e.,

$$u^{(k+1)}(u^{(k)}) = 0, \quad 0 \leq k \leq N-1, \quad u^{(0)} = u. \quad (5.59)$$

Using the relations (5.38) – (5.41), one can show that, in this case, the right hand sides of the higher-stage Noether identities (5.32) also equal zero [6]. It follows that the summand G_{r_k} of each cocycle Δ_{r_k} (5.30) is δ_{k-1} -closed. Then its summand h_{r_k} also is δ_{k-1} -closed and, consequently, δ_{k-2} -closed. Hence it is δ_{k-1} -exact by virtue of Condition 5.5. Therefore, Δ_{r_k} contains only the term G_{r_k} linear in antifields. \square

It follows at once from the equalities (5.59) that the higher-stage gauge operator

$$u_{\text{HS}} = \mathbf{u} - u = u^{(1)} + \dots + u^{(N)}$$

is nilpotent, and

$$\mathbf{u}(\mathbf{u}) = u(\mathbf{u}). \quad (5.60)$$

Therefore, the nilpotency condition for the BRST operator \mathbf{b} (5.58) takes a form

$$\mathbf{b}(\mathbf{b}) = (u + \gamma)(\mathbf{u}) + (u + u_{\text{HS}} + \gamma)(\gamma) = 0. \quad (5.61)$$

Let us denote

$$\begin{aligned}\gamma^{(0)} &= 0, \\ \gamma^{(k)} &= \gamma_{(2)}^{(k)} + \dots + \gamma_{(k+1)}^{(k)}, \quad k = 1, \dots, N+1, \\ \gamma_{(i)}^{r_{k-1}} &= \sum_{k_1 + \dots + k_i = k+1-i} \left(\sum_{0 \leq |\Lambda_{k_j}|} \gamma_{(i)r_{k_1}, \dots, r_{k_i}}^{r_{k-1}, \Lambda_{k_1}, \dots, \Lambda_{k_i}} c_{\Lambda_{k_1}}^{r_{k_1}} \dots c_{\Lambda_{k_i}}^{r_{k_i}} \right), \\ \gamma^{(N+2)} &= 0,\end{aligned}$$

where $\gamma_{(i)}^{(k)}$ are terms of polynomial degree $2 \leq i \leq k+1$ in ghosts. Then the nilpotent property (5.61) of \mathbf{b} falls into a set of equalities

$$u^{(k+1)}(u^{(k)}) = 0, \quad 0 \leq k \leq N-1, \quad (5.62)$$

$$(u + \gamma_{(2)}^{(k+1)})(u^{(k)}) + u_{\text{HS}}(\gamma_{(2)}^{(k)}) = 0, \quad 0 \leq k \leq N+1, \quad (5.63)$$

$$\gamma_{(i)}^{(k+1)}(u^{(k)}) + u(\gamma_{(i-1)}^{(k)}) + u_{\text{HS}}(\gamma_{(i)}^{(k)}) + \quad (5.64)$$

$$\sum_{2 \leq m \leq i-1} \gamma_{(m)}(\gamma_{(i-m+1)}^{(k)}) = 0, \quad i-2 \leq k \leq N+1,$$

of ghost polynomial degree 1, 2 and $3 \leq i \leq N+3$, respectively.

The equalities (5.62) are exactly the gauge symmetry conditions (5.59) in Theorem 5.11.

The equality (5.63) for $k=0$ reads

$$(u + \gamma^{(1)})(u) = 0, \quad \sum_{0 \leq |\Lambda|} (d_{\Lambda}(u^A) \partial_A^{\Lambda} u^B + d_{\Lambda}(\gamma^r) u_r^{B, \Lambda}) = 0. \quad (5.65)$$

It takes a form of the Lie antibracket

$$[u, u] = -2\gamma^{(1)}(u) = -2 \sum_{0 \leq |\Lambda|} d_{\Lambda}(\gamma^r) u_r^{B, \Lambda} \partial_B \quad (5.66)$$

of the odd gauge symmetry u . Its right-hand side is non-linear in ghosts. Following Remark 5.10, we treat it as a generalized gauge symmetry factorizing through the gauge symmetry u . Thus, we come to the following.

Theorem 5.12. *The gauge operator (5.42) admits the nilpotent extension (5.58) only if the Lie antibracket of the odd gauge symmetry u (5.48) is a generalized gauge symmetry factorizing through u .*

The equalities (5.63) – (5.64) for $k=1$ take a form

$$(u + \gamma_{(2)}^{(2)})(u^{(1)}) + u^{(1)}(\gamma^{(1)}) = 0, \quad (5.67)$$

$$\gamma_{(3)}^{(2)}(u^{(1)}) + (u + \gamma^{(1)})(\gamma^{(1)}) = 0. \quad (5.68)$$

In particular, if a Lagrangian system is irreducible, i.e., $u^{(k)} = 0$ and $\mathbf{u} = u$, the BRST operator reads

$$\mathbf{b} = u + \gamma^{(1)} = u^A \partial_A + \gamma^r \partial_r = \sum_{0 \leq |\Lambda|} u_r^{A, \Lambda} c_{\Lambda}^r \partial_A + \sum_{0 \leq |\Lambda|, |\Xi|} \gamma_{pq}^{r, \Lambda, \Xi} c_{\Lambda}^p c_{\Xi}^q \partial_r. \quad (5.69)$$

In this case, the nilpotency conditions (5.67) – (5.68) are reduced to the equality

$$(u + \gamma^{(1)})(\gamma^{(1)}) = 0. \quad (5.70)$$

Furthermore, let a gauge symmetry u be affine in fields s^A and their jets. It follows from the nilpotency condition (5.65) that the BRST term $\gamma^{(1)}$ is independent of original fields and their jets. Then the relation (5.70) takes a form of the Jacobi identity

$$\gamma^{(1)}(\gamma^{(1)}) = 0 \quad (5.71)$$

for coefficient functions $\gamma_{pq}^{r,\Lambda,\Xi}(x)$ in the Lie antibracket (5.66).

The relations (5.66) and (5.71) motivate us to think of the equalities (5.63) – (5.64) in a general case of reducible gauge symmetries as being *sui generis* generalized commutation relations and Jacobi identities of gauge symmetries, respectively [35, 36]. Based on Theorem 5.12, we therefore say that non-trivial gauge symmetries are algebraically closed (in the terminology of [37]) if the gauge operator \mathbf{u} (5.43) admits the nilpotent BRST extension \mathbf{b} (5.58).

Example 5.11. A Lagrangian system is called abelian if its gauge symmetry u is abelian and the higher-stage gauge symmetries are independent of original fields, i.e., if $u(\mathbf{u}) = 0$. It follows from the relation (5.60) that, in this case, the gauge operator itself is the BRST operator $\mathbf{u} = \mathbf{b}$. In particular, a Lagrangian system with a variationally trivial Lagrangian (Remark 5.9) is abelian, and \mathbf{u} (5.57) is the BRST operator. The topological BF theory exemplifies a reducible abelian Lagrangian system (Section 6.4).

5.5 Lagrangian BRST theory

The DBGA $P_\infty^*\{N\}$ (5.37) is a particular field-antifield theory of the following type [3, 9, 37].

Let us consider a pull-back composite bundle

$$W = Z \times_{\underset{X}{Z}'} Z' \rightarrow Z \rightarrow X$$

where $Z' \rightarrow X$ is a vector bundle. Let us regard it as a graded vector bundle over Z possessing only odd part. The density-dual \overline{VW} of the vertical tangent bundle VW of $W \rightarrow X$ is a graded vector bundle

$$\overline{VW} = ((\overline{Z'} \oplus_{\underset{Z}{Z}} V^*Z) \otimes_{\underset{Z}{Z}} \wedge^n T^*X) \oplus_{\underset{Y}{Y}} Z'$$

over Z (cf. (5.7)). Let us consider the DBGA $\mathcal{P}_\infty^*[\overline{VW}; Z]$ (5.8) with the local generating basis

$$(z^a, \overline{z}_a), \quad [\overline{z}_a] = ([z^a] + 1) \bmod 2.$$

Its elements z^a and \overline{z}_a are called fields and antifields, respectively.

Graded densities of this DBGA are endowed with the antibracket

$$\{\mathfrak{L}\omega, \mathfrak{L}'\omega\} = \left[\frac{\overleftarrow{\delta} \mathfrak{L}}{\delta \overline{z}_a} \frac{\delta \mathfrak{L}'}{\delta z^a} + (-1)^{[\mathfrak{L}']([\mathfrak{L}'] + 1)} \frac{\overleftarrow{\delta} \mathfrak{L}'}{\delta \overline{z}_a} \frac{\delta \mathfrak{L}}{\delta z^a} \right] \omega. \quad (5.72)$$

With this antibracket, one associates to any (even) Lagrangian $\mathfrak{L}\omega$ the odd vertical graded derivations

$$v_{\mathfrak{L}} = \overleftarrow{\mathcal{E}}^a \partial_a = \frac{\overleftarrow{\delta} \mathfrak{L}}{\delta \overline{z}_a} \frac{\partial}{\partial z^a}, \quad (5.73)$$

$$\overline{v}_{\mathfrak{L}} = \overleftarrow{\partial}^a \mathcal{E}_a = \frac{\overleftarrow{\partial}}{\partial \overline{z}_a} \frac{\delta \mathfrak{L}}{\delta z^a}, \quad (5.74)$$

$$\vartheta_{\mathfrak{L}} = v_{\mathfrak{L}} + \overline{v}_{\mathfrak{L}}^l = (-1)^{[a]+1} \left(\frac{\delta \mathfrak{L}}{\delta \overline{z}^a} \frac{\partial}{\partial z_a} + \frac{\delta \mathfrak{L}}{\delta z^a} \frac{\partial}{\partial \overline{z}_a} \right), \quad (5.75)$$

such that

$$\vartheta_{\mathfrak{L}}(\mathfrak{L}'\omega) = \{\mathfrak{L}\omega, \mathfrak{L}'\omega\}.$$

Theorem 5.13. *The following conditions are equivalent.*

(i) *The antibracket of a Lagrangian $\mathfrak{L}\omega$ is d_H -exact, i.e.,*

$$\{\mathfrak{L}\omega, \mathfrak{L}\omega\} = 2 \frac{\overleftarrow{\delta} \mathfrak{L}}{\delta \bar{z}_a} \frac{\delta \mathfrak{L}}{\delta z^a} \omega = d_H \sigma. \quad (5.76)$$

(ii) *The graded derivation v (5.73) is a variational symmetry of a Lagrangian $\mathfrak{L}\omega$.*

(iii) *The graded derivation \bar{v} (5.74) is a variational symmetry of $\mathfrak{L}\omega$.*

(iv) *The graded derivation $\vartheta_{\mathfrak{L}}$ (5.75) is nilpotent.*

Proof. By virtue of the first variational formula (4.15), conditions (ii) and (iii) are equivalent to condition (i). The equality (5.76) is equivalent to that the odd density $\overleftarrow{\mathcal{E}}^a \mathcal{E}_a \omega$ is variationally trivial. Replacing right variational derivatives $\overleftarrow{\mathcal{E}}^a$ with $(-1)^{[a]+1} \mathcal{E}^a$, we obtain

$$2 \sum_a (-1)^{[a]} \mathcal{E}^a \mathcal{E}_a \omega = d_H \sigma.$$

The variational operator acting on this relation results in the equalities

$$\begin{aligned} \sum_{0 \leq |\Lambda|} (-1)^{[a]+|\Lambda|} d_{\Lambda}(\partial_b^{\Lambda}(\mathcal{E}^a \mathcal{E}_a)) &= \\ \sum_{0 \leq |\Lambda|} (-1)^{[a]} [\eta(\partial_b \mathcal{E}^a)^{\Lambda} \mathcal{E}_{\Lambda a} + \eta(\partial_b \mathcal{E}_a)^{\Lambda} \mathcal{E}_{\Lambda}^a] &= 0, \\ \sum_{0 \leq |\Lambda|} (-1)^{[a]+|\Lambda|} d_{\Lambda}(\partial^{\Lambda b}(\mathcal{E}^a \mathcal{E}_a)) &= \\ \sum_{0 \leq |\Lambda|} (-1)^{[a]} [\eta(\partial^b \mathcal{E}^a)^{\Lambda} \mathcal{E}_{\Lambda a} + \eta(\partial^b \mathcal{E}_a)^{\Lambda} \mathcal{E}_{\Lambda}^a] &= 0. \end{aligned}$$

Due to the identity

$$(\delta \circ \delta)(L) = 0, \quad \eta(\partial_B \mathcal{E}_A)^{\Lambda} = (-1)^{[A][B]} \partial_A^{\Lambda} \mathcal{E}_B,$$

we obtain

$$\begin{aligned} \sum_{0 \leq |\Lambda|} (-1)^{[a]} [(-1)^{[b]([a]+1)} \partial^{\Lambda a} \mathcal{E}_b \mathcal{E}_{\Lambda a} + (-1)^{[b][a]} \partial_a^{\Lambda} \mathcal{E}_b \mathcal{E}_{\Lambda}^a] &= 0, \\ \sum_{0 \leq |\Lambda|} (-1)^{[a]+1} [(-1)^{([b]+1)([a]+1)} \partial^{\Lambda a} \mathcal{E}^b \mathcal{E}_{\Lambda a} + (-1)^{([b]+1)[a]} \partial_a^{\Lambda} \mathcal{E}^b \mathcal{E}_{\Lambda}^a] &= 0 \end{aligned}$$

for all \mathcal{E}_b and \mathcal{E}^b . This is exactly condition (iv). \square

The equality (5.76) is called the classical master equation. For instance, any variationally trivial Lagrangian satisfies the master equation. A solution of the master equation (5.76) is called non-trivial if both the derivations (5.73) and (5.74) do not vanish.

Being an element of the DBGA $\mathcal{P}_{\infty}^*\{N\}$ (5.37), an original Lagrangian L obeys the master equation (5.76) and yields the graded derivations $v_L = 0$ (5.73) and $\bar{v}_L = \bar{\delta}$ (5.74), i.e., it is a trivial solution of the master equation.

The graded derivations (5.73) – (5.74) associated to the extended Lagrangian L_e (5.52) are extensions

$$v_e = \mathbf{u} + \frac{\overleftarrow{\delta} \mathcal{L}_1^*}{\overleftarrow{\delta} \bar{s}_A} \frac{\partial}{\partial s^A} + \sum_{0 \leq k \leq N} \frac{\overleftarrow{\delta} \mathcal{L}_1^*}{\overleftarrow{\delta} \bar{c}_{r_k}} \frac{\partial}{\partial c^{r_k}},$$

$$\bar{v}_e = \delta_{\text{KT}} + \frac{\overleftarrow{\partial}}{\overleftarrow{\delta} \bar{s}_A} \frac{\delta \mathcal{L}_1}{\delta s^A}$$

of the gauge and Koszul – Tate operators, respectively. However, the Lagrangian L_e need not satisfy the master equation. Therefore, let us consider its extension

$$L_E = L_e + L' = L + L_1 + L_2 + \dots \quad (5.77)$$

by means of even densities L_i , $i \geq 2$, of zero antifield number and polynomial degree i in ghosts. The corresponding graded derivations (5.73) – (5.74) read

$$v_E = v_e + \frac{\overleftarrow{\delta} \mathcal{L}'}{\overleftarrow{\delta} \bar{s}_A} \frac{\partial}{\partial s^A} + \sum_{0 \leq k \leq N} \frac{\overleftarrow{\delta} \mathcal{L}'}{\overleftarrow{\delta} \bar{c}_{r_k}} \frac{\partial}{\partial c^{r_k}}, \quad (5.78)$$

$$\bar{v}_E = \bar{v}_e + \frac{\overleftarrow{\partial}}{\overleftarrow{\delta} \bar{s}_A} \frac{\delta \mathcal{L}'}{\delta s^A} + \sum_{0 \leq k \leq N} \frac{\overleftarrow{\partial}}{\overleftarrow{\delta} \bar{c}_{r_k}} \frac{\delta \mathcal{L}'}{\delta c^{r_k}}. \quad (5.79)$$

The Lagrangian L_E (5.77) where $L + L_1 = L_e$ is called a proper extension of an original Lagrangian L . The following is a corollary of Theorem 5.13.

Corollary 5.14. *A Lagrangian L is extended to a proper solution L_E (5.77) of the master equation only if the gauge operator \mathbf{u} (5.42) admits a nilpotent extension.*

By virtue of condition (iv) of Theorem 5.13, this nilpotent extension is the derivation $\vartheta_E = v_E + \bar{v}_E^I$ (5.75), called the KT-BRST operator. With this operator, the module of densities $P_\infty^{0,n}\{N\}$ is split into the KT-BRST complex

$$\begin{aligned} \dots \longrightarrow \mathcal{P}_\infty^{0,n}\{N\}_2 \longrightarrow \mathcal{P}_\infty^{0,n}\{N\}_1 \longrightarrow \mathcal{P}_\infty^{0,n}\{N\}_0 \longrightarrow \\ \mathcal{P}_\infty^{0,n}\{N\}^1 \longrightarrow \mathcal{P}_\infty^{0,n}\{N\}^2 \longrightarrow \dots \end{aligned} \quad (5.80)$$

Putting all ghosts zero, we obtain a cochain morphism of this complex onto the Koszul – Tate complex, extended to $\bar{\mathcal{P}}_\infty^{0,n}\{N\}$ and reversed into the cochain one. Letting all Noether antifields zero, we come to a cochain morphism of the KT-BRST complex (5.80) onto the cochain sequence (5.42), where the gauge operator is extended to the antifield-free part of the KT-BRST operator.

Theorem 5.15. *If the gauge operator \mathbf{u} (5.42) can be extended to the BRST operator \mathbf{b} (5.58), then the master equation has a non-trivial proper solution*

$$L_E = L_e + \sum_{1 \leq k \leq N} \gamma^{r_{k-1}} \bar{c}_{r_{k-1}} \omega = \quad (5.81)$$

$$L + \mathbf{b} \left(\sum_{0 \leq k \leq N} c^{r_{k-1}} \bar{c}_{r_{k-1}} \right) \omega + d_H \sigma.$$

Proof. By virtue of Theorem 5.11, if the BRST operator \mathbf{b} (5.58) exists, the densities Δ_{r_k} (5.30) contain only the terms G_{r_k} linear in antifields. It follows that the extended Lagrangian L_e (5.44) and, consequently, the Lagrangian L_E (5.81) are affine in antifields. In this case, we have

$$u^A = \overleftarrow{\delta}^A(\mathcal{L}_e), \quad u^{r_k} = \overleftarrow{\delta}^{r_k}(\mathcal{L}_e)$$

for all indices A and r_k and, consequently,

$$\mathbf{b}^A = \overleftarrow{\delta}^A(\mathcal{L}_E), \quad \mathbf{b}^{r_k} = \overleftarrow{\delta}^{r_k}(\mathcal{L}_E),$$

i.e., $\mathbf{b} = v_E$ is the graded derivation (5.78) defined by the Lagrangian L_E . Its nilpotency condition takes a form

$$\mathbf{b}(\overleftarrow{\delta}^A(\mathcal{L}_E)) = 0, \quad \mathbf{b}(\overleftarrow{\delta}^{r_k}(\mathcal{L}_E)) = 0.$$

Hence, we obtain

$$\mathbf{b}(\mathcal{L}_E) = \mathbf{b}(\overleftarrow{\delta}^A(\mathcal{L}_E)\overline{s}_A + \overleftarrow{\delta}^{r_k}(\mathcal{L}_E)\overline{c}_{r_k}) = 0,$$

i.e., \mathbf{b} is a variational symmetry of L_E . Consequently, L_E obeys the master equation. \square

For instance, let a gauge symmetry u be abelian, and let the higher-stage gauge symmetries be independent of original fields, i.e., $u(\mathbf{u}) = 0$. Then $\mathbf{u} = \mathbf{b}$ and $L_E = L_e$.

The proper solution L_E (5.81) of the master equation is called the BRST extension of an original Lagrangian L .

5.6 Appendix. Noether identities of differential operators

Noether identities of a Lagrangian system in Section 4.1 are particular Noether identities of differential operators which are described in homology terms as follows [57].

Let $E \rightarrow X$ be a vector bundle, and let \mathcal{E} be a E -valued k -order differential operator on a fibre bundle $Y \rightarrow X$. It is represented by a section \mathcal{E}^a of the pull-back bundle

$$J^k Y \times E \rightarrow J^k Y$$

endowed with bundle coordinates $(x^\lambda, y_\Sigma^j, \chi^a)$, $0 \leq |\Sigma| \leq k$ [17, 30, 48].

Definition 5.16. *One says that a differential operator \mathcal{E} obeys Noether identities if there exist an r -order differential operator Φ on the pull-back bundle*

$$E_Y = Y \times_X E \rightarrow X \tag{5.82}$$

such that its restriction onto E is a linear differential operator and its kernel contains \mathcal{E} , i.e.,

$$\Phi = \sum_{0 \leq |\Lambda|} \Phi_a^\Lambda \chi_\Lambda^a, \quad \sum_{0 \leq |\Lambda|} \Phi_a^\Lambda \mathcal{E}_\Lambda^a = 0. \tag{5.83}$$

Any differential operator admits Noether identities, e.g.,

$$\Phi = \sum_{0 \leq |\Lambda|, |\Sigma|} T_{ab}^{\Lambda\Sigma} d_\Sigma \mathcal{E}^b \chi_\Lambda^a, \quad T_{ab}^{\Lambda\Sigma} = -T_{ba}^{\Sigma\Lambda}. \tag{5.84}$$

Therefore, they must be separated into the trivial and non-trivial ones.

Lemma 5.17. *One can associate to \mathcal{E} a chain complex whose boundaries vanish on $\text{Ker}\mathcal{E}$.*

Proof. Let us consider the composite graded manifold (Y, \mathfrak{A}_{E_Y}) modelled over the vector bundle $E_Y \rightarrow Y$. Let $\mathcal{S}_\infty^0[E_Y; Y]$ be the ring of graded functions on the infinite order jet manifold $J^\infty Y$ possessing the local generating basis (y^i, ε^a) of Grassmann parity $[\varepsilon^a] = 1$. It is provided with the nilpotent graded derivation

$$\bar{\delta} = \overleftarrow{\partial}_a \varepsilon^a. \quad (5.85)$$

whose definition is independent of the choice of the local generating basis. Then we have the chain complex

$$0 \leftarrow \text{Im } \delta \xleftarrow{\bar{\delta}} \mathcal{S}_\infty^0[E_Y; Y]_1 \xleftarrow{\bar{\delta}} \mathcal{S}_\infty^0[E_Y; Y]_2 \quad (5.86)$$

of graded functions of antifield number $k \leq 2$. Its one-boundaries $\bar{\delta}\Phi$, $\Phi \in \mathcal{S}_\infty^0[E_Y; Y]_2$, by very definition, vanish on $\text{Ker}\mathcal{E}$. \square

Every one-cycle

$$\Phi = \sum_{0 \leq |\Lambda|} \Phi_a^\Lambda \varepsilon_a^\Lambda \in \mathcal{S}_\infty^0[E_Y; Y]_1 \quad (5.87)$$

of the complex (5.86) defines a linear differential operator on pull-back bundle E_Y (5.82) such that it is linear on E and its kernel contains \mathcal{E} , i.e.,

$$\delta\Phi = 0, \quad \sum_{0 \leq |\Lambda|} \Phi_a^\Lambda d_\Lambda \varepsilon^a = 0. \quad (5.88)$$

In accordance with Definition 5.16, the one-cycles (5.87) define the Noether identities (5.88) of a differential operator \mathcal{E} . These Noether identities are trivial if a cycle is a boundary, i.e., it takes a form (5.84). Accordingly, non-trivial Noether identities modulo the trivial ones are associated to elements of the homology $H_1(\delta)$ of the complex (5.87).

A differential operator is called degenerate if it obeys non-trivial Noether identities.

One can say something more if the \mathcal{O}_∞^0 -module $H_1(\delta)$ is finitely generated, i.e., it possesses the following particular structure. There are elements $\Delta \in H_1(\delta)$ making up a projective $C^\infty(X)$ -module $\mathcal{C}_{(0)}$ of finite rank which, by virtue of the Serre – Swan theorem, is isomorphic to the module of sections of some vector bundle $E_0 \rightarrow X$. Let $\{\Delta^r\}$:

$$\Delta^r = \sum_{0 \leq |\Lambda|} \Delta_a^{\Lambda r} \varepsilon_a^\Lambda, \quad \Delta_a^{\Lambda r} \in \mathcal{O}_\infty^0, \quad (5.89)$$

be local bases for this $C^\infty(X)$ -module. Then every element $\Phi \in H_1(\delta)$ factorizes as

$$\Phi = \sum_{0 \leq |\Xi|} G_r^\Xi d_\Xi \Delta^r, \quad G_r^\Xi \in \mathcal{O}_\infty Y, \quad (5.90)$$

through elements of $\mathcal{C}_{(0)}$, i.e., any Noether identity (5.88) is a corollary of the Noether identities

$$\sum_{0 \leq |\Lambda|} \Delta_a^{\Lambda r} d_\Lambda \varepsilon^a = 0, \quad (5.91)$$

called complete Noether identities.

Remark 5.12. Given an integer $N \geq 1$, let E_1, \dots, E_N be vector bundles over X . Let us denote

$$\mathcal{P}_\infty^0\{N\} = \mathcal{S}_\infty^0[E_{N-1} \oplus_X \dots \oplus_X E_1 \oplus_X E_Y; Y \times_X E_0 \oplus_X \dots \oplus_X E_N]$$

if N is even and

$$\mathcal{P}_\infty^0\{N\} = \mathcal{S}_\infty^0[E_N \oplus_X \cdots \oplus_X E_1 \oplus_X E_Y; Y \times_X E_0 \oplus_X \cdots \oplus_X E_{N-1}]$$

if N is odd.

Lemma 5.18. *If the homology $H_1(\delta)$ of the complex (5.86) is finitely generated, this complex can be extended to the one-exact complex (5.93) with a boundary operator whose nilpotency conditions are equivalent to complete Noether identities.*

Proof. Let us consider the graded commutative ring $\mathcal{P}_\infty^0\{0\}$. It possesses the local generating basis $\{y^i, \varepsilon^a, \varepsilon^r\}$ of Grassmann parity $[\varepsilon^r] = 0$ and antifield number $\text{Ant}[\varepsilon^r] = 2$. This ring is provided with the nilpotent graded derivation

$$\delta_0 = \delta + \overleftarrow{\partial}_r \Delta^r. \quad (5.92)$$

Its nilpotency conditions are equivalent to the complete Noether identities (5.91). Then the module $\mathcal{P}_\infty^0\{0\}_{\leq 3}$ of graded functions of antifield number ≤ 3 is decomposed into the chain complex

$$0 \leftarrow \text{Im } \delta \xleftarrow{\delta} \mathcal{S}_\infty^0[E_Y; Y]_1 \xleftarrow{\delta_0} \mathcal{P}_\infty^0\{0\}_2 \xleftarrow{\delta_0} \mathcal{P}_\infty^0\{0\}_3. \quad (5.93)$$

Let $H_*(\delta_0)$ denote its homology. We have

$$H_0(\delta_0) = H_0(\delta) = 0.$$

Furthermore, any one-cycle Φ up to a boundary takes the form (5.90) and, therefore, it is a δ_0 -boundary

$$\Phi = \sum_{0 \leq |\Sigma|} G_r^\Xi d_\Xi \Delta^r = \delta_0 \left(\sum_{0 \leq |\Sigma|} G_r^\Xi \varepsilon_\Xi^r \right).$$

Hence, $H_1(\delta_0) = 0$, i.e., the complex (5.93) is one-exact. \square

Let us consider the second homology $H_2(\delta_0)$ of the complex (5.93). Its two-chains read

$$\Phi = G + H = \sum_{0 \leq |\Lambda|} G_r^\Lambda \varepsilon_\Lambda^r + \sum_{0 \leq |\Lambda|, |\Sigma|} H_{ab}^{\Lambda\Sigma} \varepsilon_\Lambda^a \varepsilon_\Sigma^b. \quad (5.94)$$

Its two-cycles define the first-stage Noether identities

$$\delta_0 \Phi = 0, \quad \sum_{0 \leq |\Lambda|} G_r^\Lambda d_\Lambda \Delta^r + \delta H = 0. \quad (5.95)$$

Conversely, let the equality (5.95) hold. Then it is a cycle condition of the two-chain (5.94). The first-stage Noether identities (5.95) are trivial either if a two-cycle Φ (5.94) is a boundary or its summand G vanishes on $\text{Ker } \mathcal{E}$.

Lemma 5.19. *First-stage Noether identities can be identified with nontrivial elements of the homology $H_2(\delta_0)$ iff any δ -cycle $\Phi \in \mathcal{S}_\infty^0[E_Y; Y]_2$ is a δ_0 -boundary.*

Proof. The proof is similar to that of Lemma 5.3 [57]. \square

A degenerate differential operator is called reducible if there exist non-trivial first-stage Noether identities.

If the condition of Lemma 5.19 is satisfied, let us assume that non-trivial first-stage Noether identities are finitely generated as follows. There exists a graded projective $C^\infty(X)$ -module $\mathcal{C}_{(1)} \subset H_2(\delta_0)$ of finite rank possessing a local basis $\Delta_{(1)}$:

$$\Delta^{r_1} = \sum_{0 \leq |\Lambda|} \Delta_r^{\Lambda r_1} \varepsilon_\Lambda^r + h^{r_1},$$

such that any element $\Phi \in H_2(\delta_0)$ factorizes as

$$\Phi = \sum_{0 \leq |\Xi|} \Phi_{r_1}^\Xi d_\Xi \Delta^{r_1} \quad (5.96)$$

through elements of $\mathcal{C}_{(1)}$. Thus, all non-trivial first-stage Noether identities (5.95) result from the equalities

$$\sum_{0 \leq |\Lambda|} \Delta_r^{\Lambda r_1} d_\Lambda \Delta^r + \delta h^{r_1} = 0, \quad (5.97)$$

called the complete first-stage Noether identities.

Lemma 5.20. *If non-trivial first-stage Noether identities are finitely generated, the one-exact complex (5.93) is extended to the two-exact one (5.99) with a boundary operator whose nilpotency conditions are equivalent to complete Noether and first-stage Noether identities.*

Proof. By virtue of the Serre – Swan theorem, the module $\mathcal{C}_{(1)}$ is isomorphic to a module of sections of some vector bundle $E_1 \rightarrow X$. Let us consider the ring $\mathcal{P}_\infty^0\{1\}$ of graded functions on $J^\infty Y$ possessing the local generating bases $\{y^i, \varepsilon^a, \varepsilon^r, \varepsilon^{r_1}\}$ of Grassmann parity $[\varepsilon^{r_1}] = 1$ and antifield number $\text{Ant}[\varepsilon^{r_1}] = 3$. It can be provided with the nilpotent graded derivation

$$\delta_1 = \delta_0 + \overleftarrow{\partial}_{r_1} \Delta^{r_1}. \quad (5.98)$$

Its nilpotency conditions are equivalent to the complete Noether identities (5.91) and the complete first-stage Noether identities (5.97). Then the module $\mathcal{P}_\infty^0\{1\}_{\leq 4}$ of graded functions of antifield number ≤ 4 is decomposed into the chain complex

$$0 \leftarrow \text{Im } \delta \xleftarrow{\delta} \mathcal{S}_\infty[E_Y; Y]_1 \xleftarrow{\delta_0} \mathcal{P}_\infty^0\{0\}_2 \xleftarrow{\delta_1} \mathcal{P}_\infty^0\{1\}_3 \xleftarrow{\delta_1} \mathcal{P}_\infty^0\{1\}_4. \quad (5.99)$$

Let $H_*(\delta_1)$ denote its homology. It is readily observed that

$$H_0(\delta_1) = H_0(\delta) = 0, \quad H_1(\delta_1) = H_1(\delta_0) = 0.$$

By virtue of the expression (5.96), any two-cycle of the complex (5.99) is a boundary

$$\Phi = \sum_{0 \leq |\Xi|} \Phi_{r_1}^\Xi d_\Xi \Delta^{r_1} = \delta_1 \left(\sum_{0 \leq |\Xi|} \Phi_{r_1}^\Xi \varepsilon_\Xi^{r_1} \right).$$

It follows that $H_2(\delta_1) = 0$, i.e., the complex (5.99) is two-exact. \square

If the third homology $H_3(\delta_1)$ of the complex (5.99) is not trivial, its elements correspond to second-stage Noether identities, and so on. Iterating the arguments, we come to the following.

A degenerate differential operator \mathcal{E} is called N -stage reducible if it admits finitely generated non-trivial N -stage Noether identities, but no non-trivial $(N + 1)$ -stage ones. It is characterized as follows [57].

- There are graded vector bundles E_0, \dots, E_N over X , and the graded commutative ring $\mathcal{S}_\infty^0[E_Y; Y]$ is enlarged to the graded commutative ring $\overline{\mathcal{P}}_\infty^0\{N\}$ with the local generating basis

$$(y^i, \varepsilon^a, \varepsilon^r, \varepsilon^{r_1}, \dots, \varepsilon^{r_N})$$

of Grassmann parity $[\varepsilon^{r_k}] = (k + 1) \bmod 2$ and antifield number $\text{Ant}[\varepsilon_\Lambda^{r_k}] = k + 2$.

- The graded commutative ring $\overline{\mathcal{P}}_\infty^0\{N\}$ is provided with the nilpotent right graded derivation

$$\begin{aligned} \delta_{\text{KT}} = \delta_N = \delta_0 + \sum_{1 \leq k \leq N} \overleftarrow{\partial}_{r_k} \Delta^{r_k}, \\ \Delta^{r_k} = \sum_{0 \leq |\Lambda|} \Delta_{r_{k-1}}^{\Lambda r_k} \varepsilon_\Lambda^{r_{k-1}} + \sum_{0 \leq \Sigma, 0 \leq \Xi} (h_{ar_{k-2}}^{\Xi \Sigma r_k} \varepsilon_\Xi^a \varepsilon_\Sigma^{r_{k-2}} + \dots), \end{aligned} \quad (5.100)$$

of antifield number -1.

- With this graded derivation, the module $\mathcal{P}_\infty^0\{N\}_{\leq N+3}$ of graded functions of antifield number $\leq (N + 3)$ is decomposed into the exact Koszul – Tate complex

$$\begin{aligned} 0 \leftarrow \text{Im } \delta \xleftarrow{\delta} \mathcal{S}_\infty^0[E_Y; Y]_1 \xleftarrow{\delta_0} \mathcal{P}_\infty^0\{0\}_2 \xleftarrow{\delta_1} \mathcal{P}_\infty^0\{1\}_3 \cdots \\ \xleftarrow{\delta_{N-1}} \mathcal{P}_\infty^0\{N-1\}_{N+1} \xleftarrow{\delta_{\text{KT}}} \mathcal{P}_\infty^0\{N\}_{N+2} \xleftarrow{\delta_{\text{KT}}} \mathcal{P}_\infty^0\{N\}_{N+3}, \end{aligned} \quad (5.101)$$

which satisfies the following homology regularity condition.

Condition 5.21. Any $\delta_{k < N-1}$ -cycle

$$\Phi \in \mathcal{P}_\infty^0\{k\}_{k+3} \subset \mathcal{P}_\infty^0\{k+1\}_{k+3}$$

is a δ_{k+1} -boundary.

- The nilpotentness $\delta_{\text{KT}}^2 = 0$ of the Koszul – Tate operator (5.100) is equivalent to the complete non-trivial Noether identities (5.91) and the complete non-trivial $(k \leq N)$ -stage Noether identities

$$\begin{aligned} \sum_{0 \leq |\Lambda|} \Delta_{r_{k-1}}^{\Lambda r_k} d_\Lambda \left(\sum_{0 \leq |\Sigma|} \Delta_{r_{k-2}}^{\Sigma r_{k-1}} \varepsilon_\Sigma^{r_{k-2}} \right) + \\ \delta \left(\sum_{0 \leq \Sigma, \Xi} h_{ar_{k-2}}^{\Xi \Sigma r_k} \varepsilon_\Xi^a \varepsilon_\Sigma^{r_{k-2}} \right) = 0. \end{aligned} \quad (5.102)$$

Let us study the following example of reducible Noether identities of a differential operator which is relevant to topological BF theory (Section 6.4).

Example 5.13. Let us consider the fibre bundles

$$Y = X \times \mathbb{R}, \quad E = \wedge^{n-1} TX, \quad 2 < n, \quad (5.103)$$

coordinated by (x^λ, y) and $(x^\lambda, \chi^{\mu_1 \dots \mu_{n-1}})$, respectively. We study the E -valued differential operator

$$\mathcal{E}^{\mu_1 \dots \mu_{n-1}} = -\epsilon^{\mu_1 \dots \mu_{n-1}} y_\mu, \quad (5.104)$$

where ϵ is the Levi – Civita symbol. It defines the first order differential equation

$$d_H y = 0 \quad (5.105)$$

on the fibre bundle Y (5.103).

Putting

$$E_Y = \mathbb{R} \times_X^{n-1} TX,$$

let us consider the graded commutative ring $\mathcal{S}_\infty^*[E_Y; Y]$ of graded functions on $J^\infty Y$. It possesses the local generating basis $(y, \varepsilon^{\mu_1 \dots \mu_{n-1}})$ of Grassmann parity $[\varepsilon^{\mu_1 \dots \mu_{n-1}}] = 1$ and antifield number $\text{Ant}[\varepsilon^{\mu_1 \dots \mu_{n-1}}] = 1$. With the nilpotent derivation

$$\bar{\delta} = \frac{\overleftarrow{\partial}}{\partial \varepsilon^{\mu_1 \dots \mu_{n-1}}} \mathcal{E}^{\mu_1 \dots \mu_{n-1}},$$

we have the complex (5.86). Its one-chains read

$$\Phi = \sum_{0 \leq |\Lambda|} \Phi_{\mu_1 \dots \mu_{n-1}}^\Lambda \varepsilon_\Lambda^{\mu_1 \dots \mu_{n-1}},$$

and the cycle condition $\bar{\delta}\Phi = 0$ takes a form

$$\Phi_{\mu_1 \dots \mu_{n-1}}^\Lambda \mathcal{E}_\Lambda^{\mu_1 \dots \mu_{n-1}} = 0. \quad (5.106)$$

This equality is satisfied iff

$$\Phi_{\mu_1 \dots \mu_{n-1}}^{\lambda_1 \dots \lambda_k} \varepsilon^{\mu_1 \dots \mu_{n-1}} = -\Phi_{\mu_1 \dots \mu_{n-1}}^{\mu \lambda_2 \dots \lambda_k} \varepsilon^{\lambda_1 \mu_1 \dots \mu_{n-1}}.$$

It follows that Φ factorizes as

$$\Phi = \sum_{0 \leq |\Xi|} G_{\nu_2 \dots \nu_{n-1}}^\Xi d_\Xi \Delta^{\nu_2 \dots \nu_{n-1}} \omega$$

through graded functions

$$\begin{aligned} \Delta^{\nu_2 \dots \nu_{n-1}} &= \Delta_{\alpha_1 \dots \alpha_{n-1}}^{\lambda, \nu_2 \dots \nu_{n-1}} \varepsilon_\lambda^{\alpha_1 \dots \alpha_{n-1}} = \\ &\delta_{\alpha_1}^\lambda \delta_{\alpha_2}^{\nu_2} \dots \delta_{\alpha_{n-1}}^{\nu_{n-1}} \varepsilon_\lambda^{\alpha_1 \dots \alpha_{n-1}} = d_{\nu_1} \varepsilon^{\nu_1 \nu_2 \dots \nu_{n-1}}, \end{aligned} \quad (5.107)$$

which provide the complete Noether identities

$$d_{\nu_1} \mathcal{E}^{\nu_1 \nu_2 \dots \nu_{n-1}} = 0. \quad (5.108)$$

They can be written in the form

$$d_H d_H y = 0. \quad (5.109)$$

The graded functions (5.107) form a basis for a projective $C^\infty(X)$ -module of finite rank which is isomorphic to the module of sections of the vector bundle

$$E_0 = \wedge^{n-2} TX.$$

Therefore, let us extend the graded commutative ring $\mathcal{S}_\infty^0[E_Y; Y]$ to that $\mathcal{P}_\infty^*\{0\}$ (see Remark 5.82) possessing the local generating basis

$$(y, \varepsilon^{\mu_1 \dots \mu_{n-1}}, \varepsilon^{\mu_2 \dots \mu_{n-1}}),$$

where $\varepsilon^{\mu_2 \dots \mu_{n-1}}$ are even Noether antifields of antifield number 2. We have the nilpotent graded derivation

$$\delta_0 = \bar{\delta} + \frac{\overleftarrow{\partial}}{\partial \varepsilon^{\mu_2 \dots \mu_{n-1}}} \Delta^{\mu_2 \dots \mu_{n-1}}$$

of $\mathcal{P}_\infty^0\{0\}$. Its nilpotency is equivalent to the complete Noether identities (5.108). Then we obtain the one-exact complex (5.93).

Iterating the arguments, let us consider the vector bundles

$$E_k = \wedge^{n-k-2} TX, \quad k = 1, \dots, n-3, \\ E_{N=n-2} = X \times \mathbb{R}$$

and the graded commutative ring $\mathcal{P}_\infty^0\{N\}$ (see Remark 5.12), possessing the local generating basis

$$(y, \varepsilon^{\mu_1 \dots \mu_{n-1}}, \varepsilon^{\mu_2 \dots \mu_{n-1}}, \dots, \varepsilon^{\mu_{n-1}}, \varepsilon)$$

of Grassmann parity

$$[\varepsilon^{\mu_{k+2} \dots \mu_{n-1}}] = k \bmod 2, \quad [\varepsilon] = n,$$

and of antifield number

$$\text{Ant}[\varepsilon^{\mu_{k+2} \dots \mu_{n-1}}] = k + 2, \quad \text{Ant}[\varepsilon] = n.$$

It is provided with the nilpotent graded derivation

$$\delta_{\text{KT}} = \delta_0 + \sum_{1 \leq k \leq n-3} \frac{\overleftarrow{\partial}}{\partial \varepsilon^{\mu_{k+2} \dots \mu_{n-1}}} + \frac{\overleftarrow{\partial}}{\partial \varepsilon} d_{\mu_{n-1}} \varepsilon^{\mu_{n-1}}, \quad (5.110) \\ \Delta^{\mu_{k+2} \dots \mu_{n-1}} = d_{\mu_{k+1}} \varepsilon^{\mu_{k+1} \mu_{k+2} \dots \mu_{n-1}},$$

of antifield number -1. Its nilpotency results from the complete Noether identities (5.108) and the equalities

$$d_{\mu_{k+2}} \Delta^{\mu_{k+2} \dots \mu_{n-1}} = 0, \quad k = 0, \dots, n-3, \quad (5.111)$$

which are the $(k+1)$ -stage Noether identities (5.102). Then the Koszul – Tate complex (5.101) reads

$$0 \leftarrow \text{Im } \bar{\delta} \xleftarrow{\bar{\delta}} \mathcal{S}_\infty^0[E_Y; Y]_1 \xleftarrow{\delta_0} \mathcal{P}_\infty^0\{0\}_2 \xleftarrow{\delta_1} \mathcal{P}_\infty^0\{1\}_3 \dots \quad (5.112) \\ \xleftarrow{\delta_{n-3}} \mathcal{P}_\infty^0\{n-3\}_{n-1} \xleftarrow{\delta_{\text{KT}}} \mathcal{P}_\infty^0\{n-2\}_n \xleftarrow{\delta_{\text{KT}}} \mathcal{P}_\infty^0\{n-2\}_{n+1}.$$

It obeys Condition 5.21 as follows.

Lemma 5.22. *Any δ_k -cycle $\Phi \in \mathcal{P}_\infty^0\{k\}_{k+3}$ up to a δ_k -boundary takes a form*

$$\Phi = \sum_{(k_1 + \dots + k_i + 3i = k+3)} \sum_{(0 \leq |\Lambda_1|, \dots, |\Lambda_i|)} G_{\mu_{k_1+2}^{\Lambda_1} \dots \mu_{n-1}^{\Lambda_i}; \dots; \mu_{k_i+2}^{\Lambda_i} \dots \mu_{n-1}^{\Lambda_i}}^{\Lambda_1 \dots \Lambda_i} \quad (5.113) \\ d_{\Lambda_1} \Delta^{\mu_{k_1+2}^{\Lambda_1} \dots \mu_{n-1}^{\Lambda_1}} \dots d_{\Lambda_i} \Delta^{\mu_{k_i+2}^{\Lambda_i} \dots \mu_{n-1}^{\Lambda_i}}, \quad k_j = -1, 0, 1, \dots, n-3,$$

where $k_j = -1$ stands for $\varepsilon^{\mu_1 \dots \mu_{n-1}}$ and

$$\Delta^{\mu_1 \dots \mu_{n-1}} = \mathcal{E}^{\mu_1 \dots \mu_{n-1}}.$$

It follows that Φ is a δ_{k+1} -boundary.

Proof. Let us choose some basis element $\varepsilon^{\mu_{k+2} \dots \mu_{n-1}}$ and denote it, simply, by ε . Let Φ contain a summand $\phi_1 \varepsilon$, linear in ε . Then the cycle condition reads

$$\delta_k \Phi = \delta_k(\Phi - \phi_1 \varepsilon) + (-1)^{[\varepsilon]} \delta_k(\phi_1) \varepsilon + \phi \Delta = 0, \quad \Delta = \delta_k \varepsilon.$$

It follows that Φ contains a summand $\psi \Delta$ such that

$$(-1)^{[\varepsilon]+1} \delta_k(\psi) \Delta + \phi \Delta = 0.$$

This equality implies the relation

$$\phi_1 = (-1)^{[\varepsilon]+1} \delta_k(\psi) \quad (5.114)$$

because the reduction conditions (5.111) involve total derivatives of Δ , but not Δ . Hence,

$$\Phi = \Phi' + \delta_k(\psi \varepsilon),$$

where Φ' contains no term linear in ε . Furthermore, let ε be even and Φ have a summand $\sum \phi_r \varepsilon^r$ polynomial in ε . Then the cycle condition leads to the equalities

$$\phi_r \Delta = -\delta_k \phi_{r-1}, \quad r \geq 2.$$

Since ϕ_1 (5.114) is δ_k -exact, then $\phi_2 = 0$ and, consequently, $\phi_{r>2} = 0$. Thus, a cycle Φ up to a δ_k -boundary contains no term polynomial in c . It reads

$$\begin{aligned} \Phi = & \sum_{(k_1 + \dots + k_i + 3i = k+3)} \sum_{(0 < |\Lambda_1|, \dots, |\Lambda_i|)} G_{\mu_{k_1+2}^1 \dots \mu_{n-1}^1; \dots; \mu_{k_i+2}^i \dots \mu_{n-1}^i}^{\Lambda_1 \dots \Lambda_i} \\ & \varepsilon_{\Lambda_1}^{\mu_{k_1+2}^1 \dots \mu_{n-1}^1} \dots \varepsilon_{\Lambda_i}^{\mu_{k_i+2}^i \dots \mu_{n-1}^i}. \end{aligned} \quad (5.115)$$

However, the terms polynomial in ε may appear under general coordinate transformations

$$\varepsilon^{\nu_{k+2} \dots \nu_{n-1}} = \det \left(\frac{\partial x^\alpha}{\partial x'^\beta} \right) \frac{\partial x'^{\nu_{k+2}}}{\partial x^{\mu_{k+2}}} \dots \frac{\partial x'^{\nu_{n-1}}}{\partial x^{\mu_{n-1}}} \varepsilon^{\mu_{k+2} \dots \mu_{n-1}}$$

of a chain Φ (5.115). In particular, Φ contains the summand

$$\sum_{k_1 + \dots + k_i + 3i = k+3} F_{\nu_{k_1+2}^1 \dots \nu_{n-1}^1; \dots; \nu_{k_i+2}^i \dots \nu_{n-1}^i} \varepsilon^{\nu_{k_1+2}^1 \dots \nu_{n-1}^1} \dots \varepsilon^{\nu_{k_i+2}^i \dots \nu_{n-1}^i},$$

which must vanish if Φ is a cycle. This takes place only if Φ factorizes through the graded densities $\Delta^{\mu_{k+2} \dots \mu_{n-1}}$ (5.110) in accordance with the expression (5.113). \square

Following the proof of Lemma 5.22, one also can show that any δ_k -cycle $\Phi \in \mathcal{P}_\infty^0\{k\}_{k+2}$ up to a boundary takes a form

$$\Phi = \sum_{0 \leq |\Lambda|} G_{\mu_{k+2} \dots \mu_{n-1}}^\Lambda d_\Lambda \Delta^{\mu_{k+2} \dots \mu_{n-1}},$$

i.e., the homology $H_{k+2}(\delta_k)$ of the complex (5.112) is finitely generated by the cycles $\Delta^{\mu_{k+2} \dots \mu_{n-1}}$.

6 Classical field models

As was mentioned above, classical field theory of even and odd fields is formulated adequately a Lagrangian theory on graded bundles [36, 60, 66]. This Section provides some examples of relevant field models.

6.1 Gauge theory on principal bundles

In classical gauge theory, gauge fields are conventionally described as principal connections on principal bundles [36, 49, 65]. We consider their first order Yang – Mills Lagrangian theory.

Principal connections on a principal bundle $\pi_P : P \rightarrow X$ with a structure Lie group G are connections on P which are equivariant with respect to the right action

$$\begin{aligned} G : G \times_X P &\longrightarrow P, \\ G : p &\rightarrow pg, \quad \pi_P(p) = \pi_P(pg), \quad p \in P, \end{aligned} \quad (6.1)$$

of a structure group G on P . In order to describe them, we follow the definition of connections on a fibre bundle $Y \rightarrow X$ as global sections of the affine jet bundle $J^1 Y \rightarrow X$ [30, 49, 69].

Let $J^1 P$ be a first order jet manifold of a principal G -bundle $P \rightarrow X$. Then connections on a principal bundle $P \rightarrow X$ are global sections $A : P \rightarrow J^1 P$ of an affine jet bundle $J^1 P \rightarrow P$. In order to describe principal connections on $P \rightarrow X$, let us consider the jet prolongation

$$G \ni g : j_x^1 p \rightarrow (j_x^1 p)g = j_x^1 (pg). \quad (6.2)$$

the action (1) of G onto $J^1 P$. Taking the quotient of an affine jet bundle $J^1 P \rightarrow P$ by G (6.2), we obtain an affine bundle

$$C = J^1 P / G \rightarrow X \quad (6.3)$$

modelled over a vector bundle

$$\overline{C} = T^* X \otimes_X V_G P \rightarrow X.$$

Hence, there is the canonical vertical splitting

$$VC = C \otimes_X \overline{C}$$

of the vertical tangent bundle VC of $C \rightarrow X$. Principal connections on a principal bundle $P \rightarrow X$ are identified with global sections of the fibre bundle $C \rightarrow X$ (6.3), called the bundle of principal connections. Given an atlas

$$\Psi_P = \{(U_\alpha, \psi_\alpha^P), \varrho_{\alpha\beta}\} \quad (6.4)$$

of a principal bundle P , the bundle of principal connections C (6.3) is endowed with bundle coordinates (x^λ, a_μ^m) possessing the transformation rule

$$\varrho(a_\mu^m) \varepsilon_m = (a_\nu^m \text{Ad}_{\varrho^{-1}}(\varepsilon_m) + R_\nu^m \varepsilon_m) \frac{\partial x^\nu}{\partial x'^\mu}.$$

If G is a matrix group, this transformation rule reads

$$\varrho(a_\mu^m) \varepsilon_m = (a_\nu^m \varrho^{-1}(\varepsilon_m) \varrho - \partial_\mu(\varrho^{-1}) \varrho) \frac{\partial x^\nu}{\partial x'^\mu}.$$

A glance at this expression shows that the bundle of principal connections C fails to be a bundle with a structure group G .

We consider first order Lagrangian theory on a fibre bundle $Y = C$ (see Example 3.1). Its structure algebra $\mathcal{S}_\infty^*[F; Y] = \mathcal{S}_\infty^*[C]$ (2.40) is the graded differential algebra $\mathcal{S}_\infty^*[C] = \mathcal{O}_\infty^*C$ (2.35) of exterior forms on jet manifolds $J^r C$ of $C \rightarrow X$. Its first order Lagrangian (3.13) is a density

$$L = \mathcal{L}\omega : J^1 C \rightarrow \wedge^n T^* X \quad (6.5)$$

on a first order jet manifold $J^1 C$ possessing the adapted coordinates $(x^\mu, a_\mu^m, a_{\lambda\mu}^m)$. The corresponding Euler – Lagrange operator (3.14) reads

$$\mathcal{E}_L = \mathcal{E}_m^\mu \theta_\mu^m \wedge \omega = (\partial_m^\mu - d_\lambda \partial_m^{\lambda\mu}) \mathcal{L} \theta_\mu^m \wedge \omega. \quad (6.6)$$

Its kernel defines the Euler – Lagrange equation

$$\mathcal{E}_m^\mu = (\partial_m^\mu - d_\lambda \partial_m^{\lambda\mu}) \mathcal{L} = 0. \quad (6.7)$$

In classical gauge theory, gauge transformations are defined as vertical principal automorphisms of a principal bundle P which are equivariant with respect to the action (6.1) of a structure group G i.e.,

$$\Phi_P(pg) = \Phi_P(p)g, \quad g \in G, \quad p \in P. \quad (6.8)$$

In order to describe gauge symmetries of gauge theory on a principal bundle P , it is sufficient to consider (local) one-parameter groups of principal automorphisms of P and their infinitesimal generators which are G -invariant projectable vector fields ξ on P , called the principal vector fields. They are represented by sections of the quotient

$$V_G P = VP/G \quad (6.9)$$

of the vertical tangent bundle VP of $P \rightarrow X$ with respect to the tangent prolongation of the action (1) of G on P . It is a P -associated bundle whose typical fibre is a right Lie algebra \mathfrak{g}_r of G subject to the adjoint representation of a structure group G . Therefore, $V_G P$ (6.9) is called the Lie algebra bundle. Given the bundle atlas Ψ_P (6.4) of P , the Lie algebra bundle $V_G P$ is provided with bundle coordinates $(U_\alpha; x^\mu, \chi^m)$ with respect to the fibre frames $\{e_m = \psi_\alpha^{-1}(x)(\varepsilon_m)\}$, where $\{\varepsilon_m\}$ is a basis for the Lie algebra \mathfrak{g}_r . These coordinates possess the transition functions

$$\varrho(\chi^m) \varepsilon_m = \chi^m \text{Ad}_{\varrho^{-1}}(\varepsilon_m).$$

Then sections of the Lie algebra bundle V_Γ read

$$\xi = \xi^r e_r. \quad (6.10)$$

They form a finite-dimensional Lie $C^\infty(X)$ -algebra with respect to the Lie bracket

$$[\xi, \eta] = c_{pq}^r \xi^p \eta^q e_r, \quad (6.11)$$

where c_{pq}^r are the structure constants of a Lie algebra \mathfrak{g}_r .

Any (local) one-parameter group of principal automorphism Φ_P (6.8) of a principal bundle P admits the jet prolongation $J^1 \Phi_P$ to a one-parameter group of G -equivariant automorphism of

the jet manifold J^1P which, in turn, yields a one-parameter group of principal automorphisms Φ_C of the bundle of principal connections C (6.3). Its infinitesimal generator is a vector field on C . As a consequence, any principal vector field ξ (6.10) yields a principal vector field

$$u_\xi = (\partial_\mu \xi^r + c_{pq}^r a_\mu^p \xi^q) \partial_r^\mu. \quad (6.12)$$

on C [49].

A glance at the expression (6.12) shows that one can think of the principal vector fields u_ξ as being a linear first order differential operator on a vector space of sections of the Lie algebra bundle $V_G C$ (6.9) with values in a vector space of vertical vector fields on the bundle of principal connections C (6.3), i.e., u_ξ (6.12) are even gauge transformations (see Remark 4.5) with even gauge parameter functions ξ (6.10). However, since gauge symmetries in the second Noether theorem 5.9) are odd, we modify the definition of gauge transformations in gauge theory in accordance with Definition 4.8 as follows.

Let us treat $V_G P \rightarrow X$ as an odd vector bundle, and let $(X, \mathfrak{A}_{V_G P})$ be the corresponding simple graded manifold. Then let us consider the composite bundle

$$V_G P \times_X C \rightarrow C \rightarrow X, \quad (6.13)$$

coordinated by (x^μ, a_μ^m, χ^r) , and the graded bundle $(X, C, \mathfrak{A}_{V_G P \times_X C})$ (4.29) modelled over this composite bundle together with the local generating basis (x^μ, a_μ^m, c^r) whose terms c^r are odd. Let

$$S_\infty^*[V_G P \times_X C; C] \quad (6.14)$$

be the DBGA (4.30) together with the monomorphisms (4.31):

$$\mathcal{O}_\infty^* C \rightarrow S_\infty^*[V_G P \times_X C; C], \quad S_\infty^*[V_G P; X] \rightarrow S_\infty^*[V_G P \times_X C; C].$$

By inspection of transition functions of the principal vector field u_ξ (6.12), one can justify the existence of an odd contact derivation $J^\infty u$ of the DBGA (6.14) generated by a generalized vector field

$$u = (c_\mu^r + c_{pq}^r a_\mu^p c^q) \partial_r^\mu \quad (6.15)$$

on a graded bundle $(X, C, \mathfrak{A}_{V_G P \times_X C})$. The graded derivation u (6.15) obviously vanishes on a subring

$$S_\infty^0[V_G P; X] \subset S_\infty^0[V_G P \times_X C; C].$$

Consequently, it is a gauge transformation of a Lagrangian system

$$\mathcal{O}_\infty^* C \subset S_\infty^*[V_G P \times_X C; C],$$

parameterized by odd ghosts c^r .

In Yang – Mills gauge theory, its Lagrangian L (6.5) is required to be invariant under the gauge transformation u (6.15), i.e., it is an exact gauge symmetry. The corresponding condition reads

$$\mathbf{L}_{J^1 u} L = 0, \quad (6.16)$$

$$J^1 u = u + (c_{\lambda\mu}^r + c_{pq}^r a_\mu^p c_\lambda^q + c_{pq}^r a_{\lambda\mu}^p c^q) \partial_r^{\lambda\mu},$$

(cf. (4.23)). In this case, the first variational formula (4.25) for the Lie derivative (6.16) takes a form

$$0 = (c_\mu^r + c_{pq}^r a_\mu^p c^q) \mathcal{E}_r^\mu + d_\lambda [(c_\mu^r + c_{pq}^r a_\mu^p c^q) \partial_r^{\lambda\mu} \mathcal{L}]. \quad (6.17)$$

It leads to the gauge invariance conditions (4.37) – (4.41) which read

$$\partial_p^{\mu\lambda} \mathcal{L} + \partial_p^{\lambda\mu} \mathcal{L} = 0, \quad (6.18)$$

$$\mathcal{E}_r^\mu + d_\lambda \partial_r^{\lambda\mu} \mathcal{L} + c_{pr}^q a_\nu^p \partial_q^{\mu\nu} \mathcal{L} = 0, \quad (6.19)$$

$$c_{pq}^r (a_\mu^p \mathcal{E}_r^\mu + d_\lambda (a_\mu^p \partial_r^{\lambda\mu} \mathcal{L})) = 0. \quad (6.20)$$

One can regard the equalities (6.18) – (6.20) as the conditions of a Lagrangian L to be gauge invariant. They are brought into the form

$$\partial_p^{\mu\lambda} \mathcal{L} + \partial_p^{\lambda\mu} \mathcal{L} = 0. \quad (6.21)$$

$$\partial_q^\mu \mathcal{L} + c_{pq}^r a_\nu^p \partial_r^{\mu\nu} \mathcal{L} = 0, \quad (6.22)$$

$$c_{pq}^r (a_\mu^p \partial_r^\mu \mathcal{L} + a_{\lambda\mu}^p \partial_r^{\lambda\mu} \mathcal{L}) = 0. \quad (6.23)$$

In order to solve these equations, let us refer to the canonical splitting of the jet manifold

$$\begin{aligned} J^1 C &= C_+ \oplus_C C_- = C_+ \oplus_C (C \times_X^2 T^* X \otimes V_G P), \\ a_{\lambda\mu}^r &= \frac{1}{2} (\mathcal{F}_{\lambda\mu}^r + \mathcal{S}_{\lambda\mu}^r) = \frac{1}{2} (a_{\lambda\mu}^r + a_{\mu\lambda}^r - c_{pq}^r a_\lambda^p a_\mu^q) + \\ &\quad \frac{1}{2} (a_{\lambda\mu}^r - a_{\mu\lambda}^r + c_{pq}^r a_\lambda^p a_\mu^q), \end{aligned} \quad (6.24)$$

and let us utilize the coordinates $(a_\mu^q, \mathcal{F}_{\lambda\mu}^r, \mathcal{S}_{\lambda\mu}^r)$ (6.24). With respect to these coordinates, the equation (6.21) reads

$$\frac{\partial \mathcal{L}}{\partial \mathcal{S}_{\mu\lambda}^p} = 0. \quad (6.25)$$

Then the equation (6.22) takes a form

$$\frac{\partial \mathcal{L}}{\partial a_\mu^q} = 0. \quad (6.26)$$

A glance at the equalities (6.25) and (6.26) shows that a gauge invariant Lagrangian factorizes through the strength coordinates \mathcal{F} (6.24). Then the equation (6.23), written as

$$c_{pq}^r \mathcal{F}_{\lambda\mu}^p \frac{\partial \mathcal{L}}{\partial \mathcal{F}_{\lambda\mu}^r} = 0,$$

means that the gauge symmetry u (6.15) of a Lagrangian L is exact. The following thus has been proved.

Lemma 6.1. *The gauge theory Lagrangian (6.5) possesses the exact gauge symmetry u (6.15) only if it factorizes through the strength coordinates \mathcal{F} (6.24).*

A corollary of this result is the well-known Utiyama theorem [16].

Theorem 6.2. *There is a unique gauge invariant quadratic first order Lagrangian, called the Yang – Mills Lagrangian,*

$$L_{\text{YM}} = \frac{1}{4} a_{pq}^G g^{\lambda\mu} g^{\beta\nu} \mathcal{F}_{\lambda\beta}^p \mathcal{F}_{\mu\nu}^q \sqrt{|g|} \omega, \quad g = \det(g_{\mu\nu}), \quad (6.27)$$

where a^G is a G -invariant bilinear form on a Lie algebra \mathfrak{g}_r and g is a world metric on X .

The Euler – Lagrange operator (6.5) of the Yang – Mills Lagrangian L_{YM} (6.27) is

$$\mathcal{E}_{\text{YM}} = \mathcal{E}_r^\mu \theta_r^\mu \wedge \omega = (\delta_r^n d_\lambda + c_{rp}^n a_\lambda^p)(a_{nq}^G g^{\mu\alpha} g^{\lambda\beta} \mathcal{F}_{\alpha\beta}^q \sqrt{|g|}) \theta_\mu^r \wedge \omega. \quad (6.28)$$

Its kernel (6.7) defines the Yang – Mills equations

$$\mathcal{E}_r^\mu = (\delta_r^n d_\lambda + c_{rp}^n a_\lambda^p)(a_{nq}^G g^{\mu\alpha} g^{\lambda\beta} \mathcal{F}_{\alpha\beta}^q \sqrt{|g|}) = 0. \quad (6.29)$$

We call a Lagrangian system $(\mathcal{S}_\infty^*[C], L_{\text{YM}})$ the Yang – Mills gauge theory.

Remark 6.1. In classical gauge theory, there are Lagrangians, e.g., the Chern – Simons one (6.68) (Section 6.3) which do not factorize through the strength coordinates \mathcal{F} , and whose gauge symmetry u (6.15) is variational, but not exact.

Since the gauge symmetry u (6.15) of the Yang – Mills Lagrangian (6.27) is exact, the first variational formula (6.17) leads to a weak conservation law

$$0 \approx d_\lambda(-u_r^\mu \partial_r^\lambda \mathcal{L}_{\text{YM}})$$

of the Noether current

$$\mathcal{J}^\lambda = -(\partial_\mu \xi^r + c_{pq}^r a_\mu^p \xi^q)(a_{rq}^G g^{\mu\alpha} g^{\lambda\beta} \mathcal{F}_{\alpha\beta}^q \sqrt{|g|}). \quad (6.30)$$

In accordance with Theorem 4.9, the Noether current (6.30) is brought into the superpotential form (4.45) which reads

$$\begin{aligned} \mathcal{J}^\lambda &= c^r \mathcal{E}_r^\mu + d_\nu(c^r \partial_r^{[\nu\mu]} \mathcal{L}_{\text{YM}}), \\ U^{\nu\mu} &= c^r a_{rq}^G g^{\nu\alpha} g^{\mu\beta} \mathcal{F}_{\alpha\beta}^q \sqrt{|g|}. \end{aligned}$$

The gauge invariance conditions (6.18) – (6.20) lead to the Noether identities which the Euler – Lagrange operator \mathcal{E}_{YM} (6.28) of the Yang – Mills Lagrangian (6.27) satisfies. These Noether identities are associated to the gauge symmetry u (6.15). In accordance with the formula (5.55), they read

$$c_{rq}^p a_\mu^q \mathcal{E}_p^\mu + d_\mu \mathcal{E}_r^\mu = 0. \quad (6.31)$$

Lemma 6.3. *The Noether identities (6.31) are non-trivial.*

Proof. Following the procedure in Section 5.2, let us consider the density dual

$$\overline{VC} = V^* C \otimes_C^n \wedge T^* X = (T^* X \otimes_X V_G P)^* \otimes_C^n \wedge T^* X \quad (6.32)$$

of the vertical tangent bundle VC of $C \rightarrow X$, and let us enlarge the differential graded algebra $\mathcal{S}_\infty^*[C]$ to the DBGA (5.8):

$$\mathcal{P}_\infty^*[\overline{VC}; C] = \mathcal{S}_\infty^*[\overline{VC}; C],$$

possessing the local generating basis (a_μ^r, \bar{a}_r^μ) where \bar{a}_r^μ are odd antifields. Providing this DBGA with the nilpotent right graded derivation

$$\bar{\delta} = \frac{\overleftarrow{\partial}}{\partial \bar{a}_r^\mu} \mathcal{E}_r^\mu,$$

let us consider the chain complex (5.10). Its one-chains

$$\Delta_r = c_{rq}^p a_\mu^q \bar{a}_r^\mu + d_\mu \bar{a}_r^\mu \quad (6.33)$$

are $\bar{\delta}$ -cycles which define the Noether identities (6.31). Clearly, they are not $\bar{\delta}$ -boundaries. Therefore, the Noether identities (6.31) are non-trivial. \square

Lemma 6.4. *The Noether identities (6.31) are complete.*

Proof. The second order Euler – Lagrange operator \mathcal{E}_{YM} (6.28) takes its values into the space of sections of the vector bundle

$$(T^*X \otimes_{V_G P}^* \otimes_X^n T^*X \rightarrow X.$$

Let Φ be a first order differential operator on this vector bundle such that

$$\Phi \circ \mathcal{E}_{\text{YM}} = 0.$$

This condition holds only if the highest derivative term of the composition $\Phi^1 \circ \mathcal{E}_{\text{YM}}^2$ of the first order derivative term Φ^1 of Φ and the second order derivative term $\mathcal{E}_{\text{YM}}^2$ of \mathcal{E}_{YM} vanishes. This is the case only of

$$\Phi^1 = \Delta_r^1 = d_\mu \bar{a}_r^\mu.$$

\square

The graded densities $\Delta_r \omega$ (6.33) constitute a local basis for a $C^\infty(X)$ -module $\mathcal{C}_{(0)}$ isomorphic to a module $\overline{V_G P}(X)$ of sections of the density dual $\overline{V_G P}$ of the Lie algebra bundle $V_G P \rightarrow X$. Let us enlarge a DBGA $\mathcal{P}_\infty^*[\overline{V_C}; C]$ to a DBGA

$$\overline{\mathcal{P}}_\infty^*\{0\} = \mathcal{S}_\infty^*[\overline{V_C}; C \times_X \overline{V_G P}]$$

possessing the local generating basis $(a_\mu^r, \bar{a}_r^\mu, \bar{c}_r)$ where \bar{c}_r are even Noether antifields.

Lemma 6.5. *The Noether identities (6.31) are irreducible.*

Proof. Providing the DBGA $\overline{\mathcal{P}}_\infty^*\{0\}$ with the nilpotent odd graded derivation

$$\delta_0 = \bar{\delta} + \frac{\overleftarrow{\partial}}{\partial \bar{c}_r} \Delta_r,$$

let us consider the chain complex (5.18). Let us assume that Φ (5.19) is a two-cycle of this complex, i.e., the relation (5.20) holds. It is readily observed that Φ obeys this relation only if its first term G is $\bar{\delta}$ -exact, i.e., the first-stage Noether identities (5.20) are trivial. \square

It follows from Lemmas 6.3 – 6.5 that Yang – Mills gauge theory is an irreducible degenerate Lagrangian theory characterized by the complete Noether identities (6.31).

Following inverse second Noether Theorem 5.9, let us consider a DBGA

$$\mathcal{P}_\infty^*\{0\} = \mathcal{S}_\infty^*[\overline{V_C} \oplus_C V_G P; C \times_X \overline{V_G P}] \quad (6.34)$$

with the local generating basis $(a_\mu^r, \bar{a}_r^\mu, c^r, \bar{c}_r)$ where c_r are odd ghosts. The gauge operator \mathbf{u} (5.43) associated to the Noether identities (6.31) reads

$$\mathbf{u} = u = (c_\mu^r + c_{pq}^r a_\mu^p c^q) \partial_r^\mu. \quad (6.35)$$

It is the odd gauge symmetry (6.15) of the Yang – Mills Lagrangian L_{YM} (6.27). The gauge operator \mathbf{u} (6.35) admits the nilpotent BRST extension (5.69):

$$\mathbf{b} = (c_\mu^r + c_{pq}^r a_\mu^p c^q) \frac{\partial}{\partial a_\mu^r} - \frac{1}{2} c_{pq}^r c^p c^q \frac{\partial}{\partial c^r},$$

which is the well-known BRST operator in Yang - Mills gauge theory [37]. Then, by virtue of Theorem 5.15, the Yang – Mills Lagrangian L_{YM} is extended to a proper solution of the master equation

$$L_E = L_{\text{YM}} + (c_\mu^r + c_{pq}^r a_\mu^p c^q) \bar{a}_r^\mu \omega - \frac{1}{2} c_{pq}^r c^p c^q \bar{c}_r \omega.$$

6.2 Gauge gravitation theory on natural bundles

Gauge transformations of Einstein's General Relativity and its extensions, including gauge gravitation theory, are general covariant transformations. These are bundle automorphisms of so-called natural bundles. Therefore classical gravitation theory can be described as a field theory on natural bundles over a four-dimensional orientable manifold X , called the world manifold [36, 58, 63].

As well known, a connection Γ on a fibre bundle $Y \rightarrow X$ defines the horizontal lift $\Gamma\tau$ onto Y of any vector field τ on X . There is the category of natural bundles [46, 73] which admit the functorial lift $\tilde{\tau}$ onto T of any vector field τ on X such that $\tau \mapsto \tilde{\tau}$ is a monomorphism of the Lie algebra of vector field on X to that on T . One can think of the lift $\tilde{\tau}$ as being an infinitesimal generator of a local one-parameter group of general covariant transformations of T .

Natural bundles are exemplified by tensor bundles over X . A frame bundle LX of linear frame in the tangent spaces to X is a natural bundle. It is a principal bundle with a structure group $GL_4 = GL^+(4, \mathbb{R})$, and all bundles associated to LX also are natural are the natural ones. The bundle

$$C_K = J^1 LX / GL_4 \quad (6.36)$$

of principal connections on LX is not associated to LX , but it also is a natural bundle [30, 49].

Dynamic variables of gauge gravitation theory are linear wold connections and pseudo-Riemannian metrics on a world manifold. Thus, it is a metric-affine gravitation theory [40, 43, 63].

Linear connections on X (henceforth world connection) are principal connections on the linear frame bundle LX of X . They are represented by sections of the bundle of linear connections C_K (6.36). This is provided with bundle coordinates $(x^\lambda, k_\lambda^\nu{}_\alpha)$ such that components $k_\lambda^\nu{}_\alpha \circ K = K_\lambda^\nu{}_\alpha$ of a section K of $C_K \rightarrow X$ are coefficient of the linear connection

$$K = dx^\lambda \otimes (\partial_\lambda + K_\lambda^\mu{}_\nu \dot{x}^\nu \partial_\mu)$$

on TX with respect to the holonomic bundle coordinates $(x^\lambda, \dot{x}^\lambda)$.

In order to describe gravity, let us assume that the linear frame bundle LX admits a Lorentz structure, i.e., reduced principal subbundles with the structure Lorentz group $SO(1, 3)$. Global sections of the corresponding quotient bundle

$$\Sigma = LX/SO(1, 3) \rightarrow X \quad (6.37)$$

are pseudo-Riemannian (henceforth world) metrics on X . This fact motivates us to treat a metric gravitational field as a Higgs field [43, 58, 63].

Thus, the total configuration space of gauge gravitation theory in the absence of matter fields is the bundle product

$$Q = \Sigma \times_X C_K \quad (6.38)$$

coordinated by $(x^\lambda, \sigma^{\alpha\beta}, k_\mu^{\alpha\beta})$.

We consider first order Lagrangian theory on the fibre bundle Q (6.38) (see Example 3.1). Its structure algebra $\mathcal{S}_\infty^*[F; Y] = \mathcal{S}_\infty^*[Q]$ (2.40) is the graded differential algebra $\mathcal{S}_\infty^*[C] = \mathcal{O}_\infty^*Q$ (2.35) of exterior forms on jet manifolds J^rQ of $Q \rightarrow X$. Its first order Lagrangian (3.13) is a density

$$L_G = \mathcal{L}_G \omega : J^1Q \rightarrow \bigwedge^n T^*X \quad (6.39)$$

on a first order jet manifold J^1Q possessing the adapted coordinates

$$(x^\lambda, \sigma^{\alpha\beta}, k_\mu^{\alpha\beta}, \sigma_\lambda^{\alpha\beta}, k_{\lambda\mu}^{\alpha\beta}).$$

The corresponding Euler – Lagrange operator (3.14) reads

$$\mathcal{E}_G = (\mathcal{E}_{\alpha\beta} d\sigma^{\alpha\beta} + \mathcal{E}^\mu_{\alpha\beta} dk_\mu^{\alpha\beta}) \wedge \omega. \quad (6.40)$$

Its kernel defines the Euler – Lagrange equations

$$\mathcal{E}_{\alpha\beta} = 0, \quad \mathcal{E}^\mu_{\alpha\beta} = 0. \quad (6.41)$$

The fibre bundle Q (6.38) is a natural bundle admitting the functorial lift

$$\begin{aligned} \tilde{\tau}_{K\Sigma} &= \tau^\mu \partial_\mu + (\sigma^{\nu\beta} \partial_\nu \tau^\alpha + \sigma^{\alpha\nu} \partial_\nu \tau^\beta) \frac{\partial}{\partial \sigma^{\alpha\beta}} + \\ &(\partial_\nu \tau^\alpha k_\mu^{\nu\beta} - \partial_\beta \tau^\nu k_\mu^{\alpha\nu} - \partial_\mu \tau^\nu k_\nu^{\alpha\beta} + \partial_{\mu\beta} \tau^\alpha) \frac{\partial}{\partial k_\mu^{\alpha\beta}} \end{aligned} \quad (6.42)$$

of vector fields τ on X [36, 49, 63]. These lifts are generators of one-dimensional groups of general covariant transformations.

A glance at the expression (6.42) shows that one can think of the vector fields $\tilde{\tau}_{K\Sigma}$ as being a linear first order differential operator on a vector space of vector fields on X with values in a vector space of vector fields on the fibre bundle Q (6.38), i.e., $\tilde{\tau}_{K\Sigma}$ (6.42) are even gauge transformations (see Remark 4.5) with even gauge parameter functions τ . By the same reasons as in Yang – Mills gauge theory, we however modify the definition of gauge transformations in gauge gravitation theory in accordance with Definition 4.8 as follows.

Let us treat the tangent bundle $TX \rightarrow X$ as an odd vector bundle, and let (X, \mathfrak{A}_{TX}) be the corresponding simple graded manifold. Then let us consider the composite bundle

$$TX \times_X Q \rightarrow Q \rightarrow X, \quad (6.43)$$

coordinated by $(x^\lambda, \sigma^{\alpha\beta}, k_\mu^\alpha{}_\beta, \dot{x}^\nu)$, and the graded bundle $(X, Q, \mathfrak{A}_{TX \times_X Q})$ (4.29) modelled over this composite bundle together with the local generating basis

$$(x^\lambda, \sigma^{\alpha\beta}, k_\mu^\alpha{}_\beta, c^\nu)$$

whose terms c^ν are odd. Let

$$S_\infty^*[TX \times_X Q; Q] \quad (6.44)$$

be the DBGA (4.30) together with the monomorphisms (4.31):

$$\mathcal{O}_\infty^* Q \rightarrow S_\infty^*[TX \times_X Q; Q], \quad S_\infty^*[TX; X] \rightarrow S_\infty^*[TX \times_X Q; Q].$$

By inspection of transition functions of the principal vector field $\tilde{\tau}_{K\Sigma}$ (6.42), one can justify the existence of an odd contact derivation $J^\infty u_G$ of the DBGA (6.44) generated by a generalized vector field

$$u_G = c^\mu \partial_\mu + (\sigma^{\nu\beta} c_\nu^\alpha + \sigma^{\alpha\nu} c_\nu^\beta) \frac{\partial}{\partial \sigma^{\alpha\beta}} + (c_\nu^\alpha k_\mu^\nu{}_\beta - c_\beta^\nu k_\mu^\alpha{}_\nu - c_\mu^\nu k_\nu^\alpha{}_\beta + c_{\mu\beta}^\alpha) \frac{\partial}{\partial k_\mu^\alpha{}_\beta} \quad (6.45)$$

on a graded bundle $(X, Q, \mathfrak{A}_{TX \times_X Q})$. The graded derivation u (6.45) obviously vanishes on a subring

$$S_\infty^0[TX; X] \subset S_\infty^0[TX \times_X Q; Q].$$

Consequently, it is a gauge transformation of a Lagrangian system

$$\mathcal{O}_\infty^* Q \subset S_\infty^*[TX \times_X Q; Q]$$

parameterized by odd ghosts c^μ .

We do not specify a gravitation Lagrangian L_G on the jet manifold $J^1 Q$, but assume that the generalized vector field (6.45) is its exact gauge symmetry. Then the Euler – Lagrange operator (6.40) of this Lagrangian obeys irreducible Noether identities

$$-(\sigma_\lambda^{\alpha\beta} + 2\sigma_\nu^{\nu\beta} \delta_\lambda^\alpha) \mathcal{E}_{\alpha\beta} - 2\sigma_\nu^{\nu\beta} d_\nu \mathcal{E}_{\lambda\beta} + (-k_{\lambda\mu}^\alpha{}_\beta - k_{\nu\mu}^\nu{}_\beta \delta_\lambda^\alpha + k_{\beta\mu}^\alpha{}_\lambda + k_{\mu\lambda}^\alpha{}_\beta) \mathcal{E}^\mu{}_\alpha{}^\beta + (-k_\mu^\nu{}_\beta \delta_\lambda^\alpha + k_\mu^\alpha{}_\lambda \delta_\beta^\nu + k_\lambda^\alpha{}_\beta \delta_\mu^\nu) d_\nu \mathcal{E}^\mu{}_\alpha{}^\beta + d_{\mu\beta} \mathcal{E}^\mu{}_\lambda{}^\beta = 0$$

[5, 36].

Remark 6.2. By analogy with Theorem 6.1, one can show that, if the first order Lagrangian L_G (6.39) does not depend on the jet coordinates $\sigma_\lambda^{\alpha\beta}$ and it possesses the exact gauge symmetry (6.45), it factorizes through the curvature terms

$$\mathcal{R}_{\lambda\mu}^\alpha{}_\beta = k_{\lambda\mu}^\alpha{}_\beta - k_{\mu\lambda}^\alpha{}_\beta + k_\lambda^\gamma{}_\beta k_\mu^\alpha{}_\gamma - k_\mu^\gamma{}_\beta k_\lambda^\alpha{}_\gamma. \quad (6.46)$$

Taking the vertical part of the generalized vector field u_G (6.45), we obtain the gauge operator $\mathbf{u} = u_G$ (5.43) and its nilpotent BRST prolongation (5.69):

$$\mathbf{b} = u^{\alpha\beta} \frac{\partial}{\partial \sigma^{\alpha\beta}} + u_\mu^\alpha{}_\beta \frac{\partial}{\partial k_\mu^\alpha{}_\beta} + u^\lambda \frac{\partial}{\partial c^\lambda} = (\sigma^{\nu\beta} c_\nu^\alpha + \sigma^{\alpha\nu} c_\nu^\beta - c^\lambda \sigma_\lambda^{\alpha\beta}) \frac{\partial}{\partial \sigma^{\alpha\beta}} + (c_\nu^\alpha k_\mu^\nu{}_\beta - c_\beta^\nu k_\mu^\alpha{}_\nu - c_\mu^\nu k_\nu^\alpha{}_\beta + c_{\mu\beta}^\alpha - c^\lambda k_{\lambda\mu}^\alpha{}_\beta) \frac{\partial}{\partial k_\mu^\alpha{}_\beta} + c_\mu^\lambda c^\mu \frac{\partial}{\partial c^\lambda},$$

but this differs from that in [39]. Accordingly, an original Lagrangian L_G is extended to a solution of the master equation

$$L_E = L_G + u^{\alpha\beta} \bar{\sigma}_{\alpha\beta} \omega + u_\mu^\alpha \bar{k}^\mu_{\alpha\beta} \omega + u^\lambda \bar{c}_\lambda \omega,$$

where $\bar{\sigma}_{\alpha\beta}$, $\bar{k}^\mu_{\alpha\beta}$ and \bar{c}_λ are the corresponding antifields.

Remark 6.3. The Hilbert – Einstein Lagrangian L_{HE} of General Relativity depends only on metric variables $\sigma^{\alpha\beta}$. It is a reduced second order Lagrangian which differs from the first order one L'_{HE} in a variationally trivial term. The gauge transformations u_G (6.45) is a variational (but not exact) symmetry of the first order Lagrangian L'_{HE} , and its vertical part

$$u_V = (\sigma^{\nu\beta} c_\nu^\alpha + \sigma^{\alpha\nu} c_\nu^\beta - c^\lambda \sigma_\lambda^{\alpha\beta}) \frac{\partial}{\partial \sigma^{\alpha\beta}}$$

is so. Then the corresponding Noether identities (5.55) take the familiar form

$$\nabla_\mu \mathcal{E}_\lambda^\mu = (d_\mu + \{\mu^\beta{}_\lambda\}) \mathcal{E}_\beta^\mu = 0,$$

where $\mathcal{E}_\lambda^\mu = \sigma^{\mu\alpha} \mathcal{E}_{\alpha\lambda}$ and

$$\{\mu^\beta{}_\lambda\} = -\frac{1}{2} \sigma^{\beta\nu} (d_\mu \sigma_{\nu\lambda} + d_\lambda \sigma_{\mu\nu} - d_\nu \sigma_{\mu\lambda}) \quad (6.47)$$

are the Christoffel symbols expressed into function $\sigma_{\alpha\beta}$ of $\sigma^{\mu\nu}$ given by the relations $\sigma^{\mu\alpha} \sigma_{\alpha\beta} = \delta^\mu_\beta$.

Since the gauge symmetry u_G (6.45) is assumed to be an exact symmetries of a metric-affine gravitation Lagrangian, let us study the corresponding conservation law. This is the energy-momentum conservation laws because the gauge symmetry u_G is not vertical, and the corresponding energy-momentum current reduces to a superpotential in accordance with Theorem 4.9) [29, 54, 63].

In view of Remark 6.2, let us assume that a gravitation Lagrangian L_G is independent of the jet variables $\sigma_\lambda^{\alpha\beta}$ of a world metric and that it factorizes through the curvature terms $\mathcal{R}_{\lambda\mu}{}^{\alpha\beta}$ (6.46). Then the following relations take place:

$$\pi^{\lambda\nu}{}_\alpha{}^\beta = -\pi^{\nu\lambda}{}_\alpha{}^\beta, \quad \pi^{\lambda\nu}{}_\alpha{}^\beta = \frac{\partial \mathcal{L}_G}{\partial k_{\lambda\nu}{}^\alpha{}_\beta}, \quad (6.48)$$

$$\frac{\partial \mathcal{L}_G}{\partial k_{\nu}{}^\alpha{}_\beta} = \pi^{\lambda\nu}{}_\alpha{}^\sigma k_{\lambda}{}^\beta{}_\sigma - \pi^{\lambda\nu}{}_\sigma{}^\beta k_{\lambda}{}^\sigma{}_\alpha. \quad (6.49)$$

Let us follow the compact notation

$$\begin{aligned} y^A &= k_\mu{}^\alpha{}_\beta, \\ u_\mu{}^\alpha{}_\beta{}^\varepsilon{}^\sigma &= \delta_\mu{}^\varepsilon \delta_\beta{}^\sigma \delta_\gamma{}^\alpha, \\ u_\mu{}^\alpha{}_\beta{}^\varepsilon &= k_\mu{}^\varepsilon{}_\beta \delta_\gamma{}^\alpha - k_\mu{}^\alpha{}_\gamma \delta_\beta{}^\varepsilon - k_\gamma{}^\alpha{}_\beta \delta_\mu{}^\varepsilon. \end{aligned}$$

Then the generalized vector field (6.45) takes a form

$$u_G = c^\lambda \partial_\lambda + (\sigma^{\nu\beta} c_\nu^\alpha + \sigma^{\alpha\nu} c_\nu^\beta) \partial_{\alpha\beta} + (u^{A\beta}{}_\alpha c_\beta^\alpha + u^{A\beta\mu}{}_\alpha c_{\beta\mu}^\alpha) \partial_A.$$

We also have the equalities

$$\begin{aligned} \pi_A^\lambda u^{A\beta\mu}{}_\alpha &= \pi^{\lambda\mu}{}_\alpha{}^\beta, \\ \pi_A^\varepsilon u^{A\beta}{}_\alpha &= -\partial^\varepsilon{}_\alpha{}^\beta \mathcal{L}_G - \pi^{\varepsilon\beta}{}_\sigma{}^\gamma k_\alpha{}^\sigma{}_\gamma. \end{aligned}$$

Let a Lagrangian L_G be invariant under general covariant transformations, i.e.,

$$\mathbf{L}_{J^1 u_G} L_G = 0.$$

Then the first variational formula (4.25) takes a form

$$\begin{aligned} 0 = & (\sigma^{\nu\beta} c_\nu^\alpha + \sigma^{\alpha\nu} c_\nu^\beta - c^\lambda \sigma_\lambda^{\alpha\beta}) \delta_{\alpha\beta} \mathcal{L}_G + \\ & (u_{\alpha}^{A\beta} c_\beta^\alpha + u_{\alpha}^{A\beta\mu} c_{\beta\mu}^\alpha - c^\lambda y_\lambda^A) \delta_A \mathcal{L}_G - \\ & d_\lambda [\pi_A^\lambda (y_\alpha^A c^\alpha - u_{\alpha}^{A\beta} c_\beta^\alpha - u_{\alpha}^{A\epsilon\beta} c_{\epsilon\beta}^\alpha) - c^\lambda \mathcal{L}_G]. \end{aligned} \quad (6.50)$$

The first variational formula (6.50) on-shell leads to the weak conservation law

$$0 \approx -d_\lambda [\pi_A^\lambda (y_\alpha^A c^\alpha - u_{\alpha}^{A\beta} c_\beta^\alpha - u_{\alpha}^{A\epsilon\beta} c_{\epsilon\beta}^\alpha) - c^\lambda \mathcal{L}_G], \quad (6.51)$$

where

$$\mathcal{J}^\lambda = \pi_A^\lambda (y_\alpha^A c^\alpha - u_{\alpha}^{A\beta} c_\beta^\alpha - u_{\alpha}^{A\epsilon\beta} c_{\epsilon\beta}^\alpha) - c^\lambda \mathcal{L}_G \quad (6.52)$$

is the energy-momentum current of the metric-affine gravity.

Due to the arbitrariness of gauge parameters c^λ , the first variational formula (6.50) falls into the set of equalities (4.37) – (4.41) which read

$$\pi^{(\lambda\epsilon}_{\gamma}{}^{\sigma)} = 0, \quad (6.53)$$

$$(u_{\gamma}^{A\epsilon\sigma} \partial_A + u_{\gamma}^{A\epsilon} \partial_A^\sigma) \mathcal{L}_G = 0, \quad (6.54)$$

$$\delta_\alpha^\beta \mathcal{L}_G + 2\sigma^{\beta\mu} \delta_{\alpha\mu} \mathcal{L}_G + u_{\alpha}^{A\beta} \delta_A \mathcal{L}_G + d_\mu (\pi_A^\mu u_{\alpha}^{A\beta}) - y_\alpha^A \pi_A^\beta = 0 \quad (6.55)$$

$$\partial_\lambda \mathcal{L}_G = 0.$$

It is readily observed that the equalities (6.53) and (6.54) hold due to the relations (6.48) and (6.49), respectively.

Substituting the term $y_\alpha^A \pi_A^\beta$ from the expression (6.55) in the energy-momentum conservation law (6.51), one brings this conservation law into the form

$$\begin{aligned} 0 \approx & -d_\lambda [2\sigma^{\lambda\mu} c^\alpha \delta_{\alpha\mu} \mathcal{L}_G + u_{\alpha}^{A\lambda} c^\alpha \delta_A \mathcal{L}_G - \pi_A^\lambda u_{\alpha}^{A\beta} c_\beta^\alpha + \\ & d_\mu (\pi^{\lambda\mu}_{\alpha}{}^{\beta} c_\beta^\alpha + d_\mu (\pi_A^\mu u_{\alpha}^{A\lambda}) c^\alpha - d_\mu (\pi^{\lambda\mu}_{\alpha}{}^{\beta} \partial_\beta c^\alpha)]. \end{aligned} \quad (6.56)$$

After separating the variational derivatives, the energy-momentum conservation law (6.56) of the metric-affine gravity takes the superpotential form

$$\begin{aligned} 0 \approx & -d_\lambda [2\sigma^{\lambda\mu} c^\alpha \delta_{\alpha\mu} \mathcal{L}_G + \\ & (k_\mu{}^\lambda{}_\gamma \delta^\mu{}_\alpha{}^\gamma \mathcal{L}_G - k_\mu{}^\sigma{}_\alpha \delta^\mu{}_\sigma{}^\lambda \mathcal{L}_G - k_\alpha{}^\sigma{}_\gamma \delta^\lambda{}_\sigma{}^\gamma \mathcal{L}_G) c^\alpha + \\ & \delta^\lambda{}_\alpha{}^\mu \mathcal{L}_G c_\mu^\alpha - d_\mu (\delta^\mu{}_\alpha{}^\lambda \mathcal{L}_G) C^\alpha + d_\mu (\pi^{\mu\lambda}{}_\alpha{}^\nu (c_\nu^\alpha - k_\sigma{}^\alpha{}_\nu c^\sigma))], \end{aligned}$$

where the energy-momentum current on the shell (6.41) reduces to the generalized Komar superpotential

$$U_G^{\mu\lambda} = \pi^{\mu\lambda}{}_\alpha{}^\nu (c_\nu^\alpha - k_\sigma{}^\alpha{}_\nu c^\sigma) \quad (6.57)$$

[29, 54]. We can rewrite this superpotential as

$$U_G^{\mu\lambda} = 2 \frac{\partial \mathcal{L}_G}{\partial \mathcal{R}_{\mu\lambda}{}^{\alpha}{}_\nu} (D_\nu c^\alpha + T_\nu{}^\alpha{}_\sigma c^\sigma),$$

where D_ν is the covariant derivative relative to a connection $k_\nu^\alpha{}_\sigma$ and

$$T_\nu^\alpha{}_\sigma = k_\nu^\alpha{}_\sigma - k_\sigma^\alpha{}_\nu$$

is its torsion.

Example 6.4. Let us consider a Hilbert – Einstein Lagrangian

$$L_{\text{HE}} = \frac{1}{2\kappa} \mathcal{R} \sqrt{-\sigma} \omega,$$

$$\mathcal{R} = \sigma^{\lambda\nu} \mathcal{R}_{\alpha\lambda}{}^\alpha{}_\nu, \quad \sigma = \det(\sigma_{\alpha\beta}),$$

in a metric-affine gravitation model. Then the generalized Komar superpotential (6.57) comes to the well-known Komar superpotential if we substitute the Levi – Civita connection $k_\nu^\alpha{}_\sigma = \{\nu^\alpha{}_\sigma\}$ (6.47).

6.3 Chern – Simons topological theory

We consider gauge theory of principal connections on a principal bundle $P \rightarrow X$ with a structure real Lie group G . In contrast with the Yang – Mills Lagrangian L_{YM} (6.27), the Lagrangian L_{CS} (6.68) of Chern – Simons topological field theory on an odd-dimensional manifold X is independent of a world metric on X . Therefore, its non-trivial gauge symmetries are wider than those of the Yang – Mills one. However, some of them become trivial if $\dim X = 3$.

Note that one usually considers a local Chern – Simons Lagrangian which is the local Chern – Simons form derived from the local transgression formula for the Chern characteristic form. A global Chern – Simons Lagrangian is well defined, but depends on a background gauge potential [13, 32, 56].

Let $P \rightarrow X$ be a principal bundle with a structure Lie group G and C the bundle of principal connections (6.3) coordinated by (x^λ, c_μ^r) (see Section 6.1). One can show [49] that the quotient bundle $J^1 P \rightarrow C$ is a principal bundle with a structure group G which is canonically isomorphic to the pull-back

$$J^1 P = P_C = C \times_X P \rightarrow C. \quad (6.58)$$

This bundle admits the canonical principal connection

$$\mathcal{A} = dx^\lambda \otimes (\partial_\lambda + a_\lambda^p e_p) + da_\lambda^r \otimes \partial_r^\lambda, \quad (6.59)$$

with the strength

$$F_{\mathcal{A}} = (da_\mu^r \wedge dx^\mu + \frac{1}{2} c_{pq}^r a_\lambda^p a_\mu^q dx^\lambda \wedge dx^\mu) \otimes e_r. \quad (6.60)$$

Let

$$I_k(\chi) = b_{r_1 \dots r_k} \chi^{r_1} \dots \chi^{r_k} \quad (6.61)$$

be a G -invariant polynomial of degree $k > 1$ on a Lie algebra \mathfrak{g}_r of G . With the strength $F_{\mathcal{A}}$ (6.60) of the canonical principal connection \mathcal{A} (6.59), one can associate to this polynomial I_k a closed $2k$ -form

$$P_{2k}(F_{\mathcal{A}}) = b_{r_1 \dots r_k} F_{\mathcal{A}}^{r_1} \wedge \dots \wedge F_{\mathcal{A}}^{r_k}, \quad 2k \leq n, \quad (6.62)$$

on a bundle of principal connections C which is invariant under automorphisms of C induced by vertical principal automorphisms of P . Given a section A of $C \rightarrow X$, the pull-back

$$P_{2k}(F_A) = A^* P_{2k}(F_A) \quad (6.63)$$

of the form $P_{2k}(F_A)$ (6.62) is a closed $2k$ -form on X where

$$\begin{aligned} F_A &= \frac{1}{2} F_{\lambda\mu}^r dx^\lambda \wedge dx^\mu \otimes e_r, \\ F_{\lambda\mu}^r &= [\partial_\lambda + A_\lambda^p e_p, \partial_\mu + A_\mu^q e_q]^r = \partial_\lambda A_\mu^r - \partial_\mu A_\lambda^r + c_{pq}^r A_\lambda^p A_\mu^q, \end{aligned} \quad (6.64)$$

is a strength of a principal connection A . One calls the $P_{2k}(F_A)$ (6.63) the characteristic form because of its following properties [20, 49].

- Every characteristic form $P_{2k}(F_A)$ (6.63) is a closed form, i.e., $dP_{2k}(F_A) = 0$;
- The difference $P_{2k}(F_A) - P_{2k}(F_{A'})$ of characteristic forms is an exact form, whenever A and A' are different principal connections on a principal bundle P .

It follows that characteristic forms $P_{2k}(F_A)$ possesses the same de Rham cohomology class $[P_{2k}(F_A)]$ for all principal connections A on P . The association

$$I_k(\chi) \rightarrow [P_{2k}(F_A)] \in H_{\text{DR}}^*(X)$$

is the well-known Weil homomorphism.

Let I_k (6.61) be a G -invariant polynomial of degree $k > 1$ on the Lie algebra \mathfrak{g}_r of G . Let $P_{2k}(F_A)$ (6.62) be the corresponding closed $2k$ -form on C and $P_{2k}(F_A)$ (6.63) its pullback onto X by means of a section A of $C \rightarrow X$. Let the same symbol $P_{2k}(F_A)$ stand for its pull-back onto C . Since $C \rightarrow X$ is an affine bundle and, consequently, the de Rham cohomology of C equals that of X , the exterior forms $P_{2k}(F_A)$ and $P_{2k}(F_A)$ possess the same de Rham cohomology class

$$[P_{2k}(F_A)] = [P_{2k}(F_A)]$$

for any principal connection A . Consequently, the exterior forms $P_{2k}(F_A)$ and $P_{2k}(F_A)$ on C differ from each other in an exact form

$$P_{2k}(F_A) - P_{2k}(F_A) = d\mathfrak{S}_{2k-1}(a, A). \quad (6.65)$$

This relation is called the transgression formula on C [36]. Its pull-back by means of a section B of $C \rightarrow X$ gives the transgression formula on a base X :

$$P_{2k}(F_B) - P_{2k}(F_A) = d\mathfrak{S}_{2k-1}(B, A).$$

For instance, let

$$c(F_A) = \det \left(\mathbf{1} + \frac{i}{2\pi} F_A \right) = 1 + c_1(F_A) + c_2(F_A) + \cdots$$

be the total Chern form on a bundle of principal connections C . Its components $c_k(F_A)$ are Chern characteristic forms on C . If

$$P_{2k}(F_A) = c_k(F_A)$$

is the characteristic Chern $2k$ -form, then $\mathfrak{S}_{2k-1}(a, A)$ (6.65) is the Chern – Simons $(2k-1)$ -form.

In particular, one can choose a local section $A = 0$. In this case, $\mathfrak{S}_{2k-1}(a, 0)$ is called the local Chern – Simons form. Let $\mathfrak{S}_{2k-1}(A, 0)$ be its pull-back onto X by means of a section A of $C \rightarrow X$. Then the Chern – Simons form $\mathfrak{S}_{2k-1}(a, A)$ (6.65) admits the decomposition

$$\mathfrak{S}_{2k-1}(a, A) = \mathfrak{S}_{2k-1}(a, 0) - \mathfrak{S}_{2k-1}(A, 0) + dK_{2k-1}. \quad (6.66)$$

The transgression formula (6.65) also yields the transgression formula

$$\begin{aligned} h_0(P_{2k}(F_A) - P_{2k}(F_A)) &= d_H(h_0\mathfrak{S}_{2k-1}(a, A)), \\ h_0\mathfrak{S}_{2k-1}(a, A) &= k \int_0^1 \mathcal{P}_{2k}(t, A) dt, \\ \mathcal{P}_{2k}(t, A) &= b_{r_1 \dots r_k} (a_{\mu_1}^{r_1} - A_{\mu_1}^{r_1}) dx^{\mu_1} \wedge \mathcal{F}^{r_2}(t, A) \wedge \dots \wedge \mathcal{F}^{r_k}(t, A), \\ \mathcal{F}^{r_j}(t, A) &= \frac{1}{2} [ta_{\lambda_j \mu_j}^{r_j} + (1-t)\partial_{\lambda_j} A_{\mu_j}^{r_j} - ta_{\mu_j \lambda_j}^{r_j} - \\ &\quad (1-t)\partial_{\mu_j} A_{\lambda_j}^{r_j} + \frac{1}{2} c_{pq}^{r_j} (ta_{\lambda_j}^p + (1-t)A_{\lambda_j}^p)(ta_{\mu_j}^q + \\ &\quad (1-t)A_{\mu_j}^q)] dx^{\lambda_j} \wedge dx^{\mu_j} \otimes e_r, \end{aligned} \quad (6.67)$$

on $J^1 C$ (where $b_{r_1 \dots r_k}$ are coefficients of the invariant polynomial (6.61)).

If $2k - 1 = \dim X$, the density (6.67) is the global Chern – Simons Lagrangian

$$L_{CS}(A) = h_0\mathfrak{S}_{2k-1}(a, A) \quad (6.68)$$

of Chern – Simons topological field theory. It depends on a background gauge field A . The decomposition (6.66) induces the decomposition

$$L_{CS}(A) = h_0\mathfrak{S}_{2k-1}(a, 0) - h_0\mathfrak{S}_{2k-1}(A, 0) + d_H h_0 K_{2k-1}, \quad (6.69)$$

where

$$L_{CS} = h_0\mathfrak{S}_{2k-1}(a, 0) \quad (6.70)$$

is a local Chern – Simons Lagrangian.

For instance, if $\dim X = 3$, the global Chern – Simons Lagrangian (6.68) reads

$$\begin{aligned} L_{CS}(A) &= \left[\frac{1}{2} h_{mn} \varepsilon^{\alpha\beta\gamma} a_\alpha^m (\mathcal{F}_{\beta\gamma}^n - \frac{1}{3} c_{pq}^n a_\beta^p a_\gamma^q) \right] \omega - \\ &\quad \left[\frac{1}{2} h_{mn} \varepsilon^{\alpha\beta\gamma} A_\alpha^m (F_{\beta\gamma}^n - \frac{1}{3} c_{pq}^n A_\beta^p A_\gamma^q) \right] \omega - \\ &\quad d_\alpha (h_{mn} \varepsilon^{\alpha\beta\gamma} a_\beta^m A_\gamma^n) \omega, \end{aligned} \quad (6.71)$$

where $\varepsilon^{\alpha\beta\gamma}$ is the skew-symmetric Levi – Civita tensor.

Since a density

$$-\mathfrak{S}_{2k-1}(A, 0) + d_H h_0 K_{2k-1}$$

is variationally trivial, the global Chern – Simons Lagrangian (6.68) possesses the same Noether identities and gauge symmetries as the local one (6.70). They are the following.

In contrast with a Yang – Mills Lagrangian, the Chern – Simons one $L_{CS}(B)$ is independent of a world metric on X . Therefore, its gauge symmetries are all G -invariant vector fields on a principal bundle P . They are identified with sections

$$\xi = \tau^\lambda \partial_\lambda + \xi^r e_r, \quad (6.72)$$

of the vector bundle

$$T_G P = TP/G \rightarrow X, \quad (6.73)$$

and yield the vector fields

$$v = \tau^\lambda \partial_\lambda + (-c_{pq}^r \xi^p a_\lambda^q + \partial_\lambda \xi^r - a_\mu^r \partial_\lambda \tau^\mu) \partial_r^\lambda \quad (6.74)$$

on a bundle of principal connections C . Sections ξ (6.72) play a role of gauge parameters.

Lemma 6.6. *Vector fields (6.74) are locally variational symmetries of the global Chern – Simons Lagrangian $L_{CS}(A)$ (6.68).*

Proof. Since $\dim X = 2k - 1$, the transgression formula (6.65) takes a form

$$P_{2k}(F_A) = d\mathfrak{S}_{2k-1}(a, A).$$

The Lie derivative $\mathbf{L}_{J^1 v}$ acting on its sides results in the equality

$$0 = d(v]d\mathfrak{S}_{2k-1}(a, A)) = d(\mathbf{L}_{J^1 v} \mathfrak{S}_{2k-1}(a, A)),$$

i.e., the Lie derivative $\mathbf{L}_{J^1 v} \mathfrak{S}_{2k-1}(a, A)$ is locally d -exact. Consequently, the horizontal form $h_0 \mathbf{L}_{J^1 v} \mathfrak{S}_{2k-1}(a, A)$ is locally d_H -exact. A direct computation shows that

$$h_0 \mathbf{L}_{J^1 v} \mathfrak{S}_{2k-1}(a, A) = \mathbf{L}_{J^1 v} (h_0 \mathfrak{S}_{2k-1}(a, A)) + d_H S.$$

It follows that the Lie derivative $\mathbf{L}_{J^1 v} L_{CS}(A)$ of the global Chern – Simons Lagrangian along any vector field v (6.74) is locally d_H -exact, i.e., this vector field is locally a variational symmetry of $L_{CS}(A)$. \square

By virtue of item (iii) of Lemma 4.6, a vertical part

$$v_V = (-c_{pq}^r \xi^p a_\lambda^q + \partial_\lambda \xi^r - a_\mu^r \partial_\lambda \tau^\mu - \tau^\mu a_{\mu\lambda}^r) \partial_r^\lambda \quad (6.75)$$

of the vector field v (6.74) also is locally a variational symmetry of $L_{CS}(A)$.

Given the fibre bundle $T_G P \rightarrow X$ (6.73), let the same symbol also stand for the pull-back of $T_G P$ onto C . Let us consider the DBGA (5.8):

$$\mathcal{P}_\infty^*[T_G P; C] = \mathcal{S}_\infty^*[T_G P; C],$$

possessing the local generating basis $(a_\lambda^r, c^\lambda, c^r)$ of even fields a_λ^r and odd ghosts c^λ, c^r . Substituting these ghosts for gauge parameters in the vector field v (6.75), we obtain the odd vertical graded derivation

$$u = (-c_{pq}^r c^p a_\lambda^q + c_\lambda^r - c_\lambda^\mu a_\mu^r - c^\mu a_{\mu\lambda}^r) \partial_r^\lambda \quad (6.76)$$

of the DBGA $\mathcal{P}_\infty^*[T_G P; C]$. This graded derivation as like as vector fields v_V (6.75) is locally a variational symmetry of the global Chern – Simons Lagrangian $L_{CS}(A)$ (6.68), i.e., the odd density $\mathbf{L}_{J^1 u}(L_{CS}(A))$ is locally d_H -exact. Hence, it is δ -closed and, consequently, d_H -exact in accordance with Corollary 3.4. Thus, the graded derivation u (6.76) is a variational symmetry and, consequently, a gauge symmetry of the global Chern – Simons Lagrangian $L_{CS}(A)$.

By virtue of the formulas (5.54) – (5.55), the corresponding Noether identities read

$$\bar{\delta} \Delta_j = -c_{ji}^r a_\lambda^i \mathcal{E}_r^\lambda - d_\lambda \mathcal{E}_j^\lambda = 0, \quad (6.77)$$

$$\bar{\delta} \Delta_\mu = -a_{\mu\lambda}^r \mathcal{E}_r^\lambda + d_\lambda (a_\mu^r \mathcal{E}_r^\lambda) = 0. \quad (6.78)$$

They are irreducible and non-trivial, unless $\dim X = 3$. Therefore, the gauge operator (5.43) is $\mathbf{u} = u$. It admits the nilpotent BRST extension (5.69) which takes a form

$$\mathbf{b} = (-c_{ji}^r c^j a_\lambda^i + c_\lambda^r - c_\lambda^\mu a_\mu^r - c^\mu a_{\mu\lambda}^r) \frac{\partial}{\partial a_\lambda^r} - \frac{1}{2} c_{ij}^r c^i c^j \frac{\partial}{\partial c^r} + c_\mu^\lambda c^\mu \frac{\partial}{\partial c^\lambda}. \quad (6.79)$$

In order to include antifields $(\bar{a}_r^\lambda, \bar{c}_r, \bar{c}_\mu)$, let us enlarge the DBGA $\mathcal{P}_\infty^*[T_G P; C]$ to the DBGA

$$\mathcal{P}_\infty^*\{0\} = \mathcal{S}_\infty^*[\overline{VC} \oplus T_G P; C \times_X \overline{T_G P}]$$

where \overline{VC} is the density dual (6.32) of the vertical tangent bundle VC of $C \rightarrow X$ and $\overline{T_G P}$ is the density dual of $T_G P \rightarrow X$ (cf. (6.34)). By virtue of Theorem 5.15, given the BRST operator \mathbf{b} (6.79), the global Chern – Simons Lagrangian $L_{\text{CS}}(A)$ (6.68) is extended to the proper solution (5.81) of the master equation which reads

$$L_E = L_{\text{CS}}(A) + (-c_{pq}^r c^p a_\lambda^q + c_\lambda^r - c_\lambda^\mu a_\mu^r - c^\mu a_{\mu\lambda}^r) \bar{a}_r^\lambda \omega - \frac{1}{2} c_{ij}^r c^i c^j \bar{c}_r \omega + c_\mu^\lambda c^\mu \bar{c}_\lambda \omega.$$

If $\dim X = 3$, the global Chern – Simons Lagrangian takes the form (6.71). Its Euler – Lagrange operator is

$$\delta L_{\text{CS}}(B) = \mathcal{E}_r^\lambda \theta_\lambda^r \wedge \omega, \quad \mathcal{E}_r^\lambda = h_{rp} \varepsilon^{\lambda\beta\gamma} \mathcal{F}_{\beta\gamma}^p.$$

A glance at the Noether identities (6.77) – (6.78) shows that they are equivalent to the Noether identities

$$\bar{\delta} \Delta_j = -c_{ji}^r a_\lambda^i \mathcal{E}_r^\lambda - d_\lambda \mathcal{E}_j^\lambda = 0, \quad (6.80)$$

$$\bar{\delta} \Delta'_\mu = \bar{\delta} \Delta_\mu + a_\mu^r \bar{\delta} \Delta_r = c^\mu \mathcal{F}_{\lambda\mu}^r \mathcal{E}_r^\lambda = 0. \quad (6.81)$$

These Noether identities define the gauge symmetry u (6.76) written in the form

$$u = (-c_{pq}^r c'^p a_\lambda^q + c_\lambda'^r + c^\mu \mathcal{F}_{\lambda\mu}^r) \partial_r^\lambda \quad (6.82)$$

where $c'^r = c^r - a_\mu^r c^\mu$. It is readily observed that, if $\dim X = 3$, the Noether identities $\bar{\delta} \Delta'_\mu$ (6.81) are trivial. Then the corresponding part $c^\mu \mathcal{F}_{\lambda\mu}^r \partial_r^\lambda$ of the gauge symmetry u (6.82) also is trivial. Consequently, the non-trivial gauge symmetry of the Chern – Simons Lagrangian (6.71) is

$$u = (-c_{pq}^r c'^p a_\lambda^q + c_\lambda'^r) \partial_r^\lambda.$$

6.4 Topological BF theory

We address the topological BF theory of two exterior forms A and B of form degree $|A| + |B| = \dim X - 1$ on a smooth manifold X [12, 36]. It is reducible degenerate Lagrangian theory which satisfies the homology regularity condition (Condition 5.5) [8]. Its dynamic variables A and B are sections of a fibre bundle

$$Y = \bigwedge^p T^* X \oplus \bigwedge^q T^* X, \quad p + q = n - 1 > 1,$$

coordinated by $(x^\lambda, A_{\mu_1 \dots \mu_p}, B_{\nu_1 \dots \nu_q})$. Without a loss of generality, let q be even and $q \geq p$. The corresponding differential graded algebra is $\mathcal{O}_\infty^* Y$ (2.35).

There are the canonical p - and q -forms

$$\begin{aligned} A &= A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \\ B &= B_{\nu_1 \dots \nu_q} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_q} \end{aligned}$$

on Y . A Lagrangian of topological BF theory reads

$$L_{\text{BF}} = A \wedge d_H B = \epsilon^{\mu_1 \dots \mu_n} A_{\mu_1 \dots \mu_p} d_{\mu_{p+1}} B_{\mu_{p+2} \dots \mu_n} \omega, \quad (6.83)$$

where ϵ is the Levi – Civita symbol. It is a reduced first order Lagrangian. Its first order Euler – Lagrange operator (3.14) is

$$\delta L = \mathcal{E}_A^{\mu_1 \dots \mu_p} dA_{\mu_1 \dots \mu_p} \wedge \omega + \mathcal{E}_B^{\nu_{p+2} \dots \nu_n} dB_{\nu_{p+2} \dots \nu_n} \wedge \omega, \quad (6.84)$$

$$\mathcal{E}_A^{\mu_1 \dots \mu_p} = \epsilon^{\mu_1 \dots \mu_n} d_{\mu_{p+1}} B_{\mu_{p+2} \dots \mu_n}, \quad (6.85)$$

$$\mathcal{E}_B^{\mu_{p+2} \dots \mu_n} = -\epsilon^{\mu_1 \dots \mu_n} d_{\mu_{p+1}} A_{\mu_1 \dots \mu_p}. \quad (6.86)$$

The corresponding Euler – Lagrange equations can be written in a form

$$d_H B = 0, \quad d_H A = 0. \quad (6.87)$$

They obey the Noether identities

$$d_H d_H B = 0, \quad d_H d_H A = 0. \quad (6.88)$$

One can regard the components $\mathcal{E}_A^{\mu_1 \dots \mu_p}$ (6.85) and $\mathcal{E}_B^{\mu_{p+2} \dots \mu_n}$ (6.86) of the Euler – Lagrange operator (6.84) as a $(\overset{p}{\wedge} T X) \otimes_X (\overset{n}{\wedge} T^* X)$ -valued differential operator on the fibre bundle $\overset{q}{\wedge} T^* X$ and a $(\overset{q}{\wedge} T X) \otimes_X (\overset{n}{\wedge} T^* X)$ -valued differential operator on the fibre bundle $\overset{p}{\wedge} T^* X$, respectively. They are of the same type as the $\overset{n-1}{\wedge} T X$ -valued differential operator (5.104) in Example 5.13 (cf. the equations (6.87) and (5.103)). Therefore, the analysis of Noether identities of the differential operators (6.85) and (6.86) is a repetition of that of Noether identities of the operator (5.104) (cf. the Noether identities (6.88) and (5.109)).

Following Example 5.13, let us consider the family of vector bundles

$$\begin{aligned} E_k &= \overset{p-k-1}{\wedge} T^* X \times_X \overset{q-k-1}{\wedge} T^* X, \quad 0 \leq k < p-1, \\ E_k &= \mathbb{R} \times_X \overset{q-p}{\wedge} T^* X, \quad k = p-1, \\ E_k &= \overset{q-k-1}{\wedge} T^* X, \quad p-1 < k < q-1, \\ E_{q-1} &= X \times \mathbb{R}. \end{aligned}$$

Let us enlarge the differential graded algebra $\mathcal{O}_\infty^* Y$ to the BGDA $\mathcal{P}_\infty^* \{q-1\}$ (5.37) which is

$$\mathcal{P}_\infty^* \{q-1\} = \mathcal{P}_\infty^* [\overline{VY} \oplus_Y E_0 \oplus \dots \oplus_Y E_{q-1} \oplus_Y \overline{E}_0 \oplus_Y \dots \oplus_Y \overline{E}_{q-1}; Y]. \quad (6.89)$$

It possesses the local generating basis

$$\begin{aligned} \{ & A_{\mu_1 \dots \mu_p}, B_{\nu_1 \dots \nu_q}, \varepsilon_{\mu_2 \dots \mu_p}, \dots, \varepsilon_{\mu_p}, \varepsilon, \xi_{\nu_2 \dots \nu_q}, \dots, \xi_{\nu_q}, \xi, \\ & \overline{A}^{\mu_1 \dots \mu_p}, \overline{B}^{\nu_1 \dots \nu_q}, \overline{\varepsilon}^{\mu_2 \dots \mu_p}, \dots, \overline{\varepsilon}^{\mu_p}, \overline{\varepsilon}, \overline{\xi}^{\nu_2 \dots \nu_q}, \dots, \overline{\xi}^{\nu_q}, \overline{\xi} \} \end{aligned}$$

of Grassmann parity

$$\begin{aligned} [\varepsilon_{\mu_k \dots \mu_p}] &= [\xi_{\nu_k \dots \nu_q}] = (k+1) \bmod 2, & [\varepsilon] &= p \bmod 2, & [\xi] &= 0, \\ [\bar{\varepsilon}^{\mu_k \dots \mu_p}] &= [\bar{\xi}^{\nu_k \dots \nu_q}] = k \bmod 2, & [\bar{\varepsilon}] &= (p+1) \bmod 2, & [\bar{\xi}] &= 1, \end{aligned}$$

of ghost number

$$\text{gh}[\varepsilon_{\mu_k \dots \mu_p}] = \text{gh}[\xi_{\nu_k \dots \nu_q}] = k, \quad \text{gh}[\varepsilon] = p+1, \quad \text{gh}[\xi] = q+1,$$

and of antifield number

$$\begin{aligned} \text{Ant}[\bar{A}^{\mu_1 \dots \mu_p}] &= \text{Ant}[\bar{B}^{\nu_{p+1} \dots \nu_q}] = 1, \\ \text{Ant}[\bar{\varepsilon}^{\mu_k \dots \mu_p}] &= \text{Ant}[\bar{\xi}^{\nu_k \dots \nu_q}] = k+1, \\ \text{Ant}[\bar{\varepsilon}] &= p, \quad \text{Ant}[\bar{\xi}] = q. \end{aligned}$$

One can show that the homology regularity condition holds (see Lemma 5.22) and that the DBGA $\mathcal{P}_\infty^*\{q-1\}$ is endowed with the Koszul – Tate operator

$$\delta_{\text{KT}} = \frac{\overleftarrow{\partial}}{\partial \bar{A}^{\mu_1 \dots \mu_p}} \mathcal{E}_A^{\mu_1 \dots \mu_p} + \frac{\overleftarrow{\partial}}{\partial \bar{B}^{\nu_1 \dots \nu_q}} \mathcal{E}_B^{\nu_1 \dots \nu_q} + \quad (6.90)$$

$$\sum_{2 \leq k \leq p} \frac{\overleftarrow{\partial}}{\partial \bar{\varepsilon}^{\mu_k \dots \mu_p}} \Delta_A^{\mu_k \dots \mu_p} + \frac{\overleftarrow{\partial}}{\partial \bar{\varepsilon}} d_{\mu_p} \bar{\varepsilon}^{\mu_p} + \quad (6.91)$$

$$\sum_{2 \leq k \leq q} \frac{\overleftarrow{\partial}}{\partial \bar{\xi}^{\nu_k \dots \nu_q}} \Delta_B^{\nu_k \dots \nu_q} + \frac{\overleftarrow{\partial}}{\partial \bar{\xi}} d_{\nu_q} \bar{\xi}^{\nu_q},$$

$$\begin{aligned} \Delta_A^{\mu_2 \dots \mu_p} &= d_{\mu_1} \bar{A}^{\mu_1 \dots \mu_p}, & \Delta_A^{\mu_{k+1} \dots \mu_p} &= d_{\mu_k} \bar{\varepsilon}^{\mu_k \mu_{k+1} \dots \mu_p}, & 2 \leq k < p, \\ \Delta_B^{\nu_2 \dots \nu_q} &= d_{\nu_1} \bar{B}^{\nu_1 \dots \nu_q}, & \Delta_B^{\nu_{k+1} \dots \nu_q} &= d_{\nu_k} \bar{\xi}^{\nu_k \nu_{k+1} \dots \nu_q}, & 2 \leq k < q. \end{aligned}$$

Its nilpotentness provides the complete Noether identities (6.87):

$$d_{\mu_1} \mathcal{E}_A^{\mu_1 \dots \mu_p} = 0, \quad d_{\nu_1} \mathcal{E}_B^{\nu_1 \dots \nu_q} = 0,$$

and the $(k-1)$ -stage ones

$$\begin{aligned} d_{\mu_k} \Delta_A^{\mu_k \dots \mu_p} &= 0, & k &= 2, \dots, p, \\ d_{\nu_k} \Delta_B^{\nu_k \dots \nu_q} &= 0, & k &= 2, \dots, q, \end{aligned}$$

(cf. (5.111)). It follows that the topological BF theory is $(q-1)$ -reducible.

Applying inverse second Noether Theorem 5.9, one obtains the gauge operator (5.43) which reads

$$\begin{aligned} \mathbf{u} &= d_{\mu_1} \varepsilon_{\mu_2 \dots \mu_p} \frac{\partial}{\partial A_{\mu_1 \mu_2 \dots \mu_p}} + d_{\nu_1} \xi_{\nu_2 \dots \nu_q} \frac{\partial}{\partial B_{\nu_1 \nu_2 \dots \nu_q}} + \\ &\quad \left[d_{\mu_2} \varepsilon_{\mu_3 \dots \mu_p} \frac{\partial}{\partial \varepsilon_{\mu_2 \mu_3 \dots \mu_p}} + \dots + d_{\mu_p} \varepsilon \frac{\partial}{\partial \varepsilon_{\mu_p}} \right] + \\ &\quad \left[d_{\nu_2} \xi_{\nu_3 \dots \nu_q} \frac{\partial}{\partial \xi_{\nu_2 \nu_3 \dots \nu_q}} + \dots + d_{\nu_q} \xi \frac{\partial}{\partial \xi_{\nu_q}} \right]. \end{aligned} \quad (6.92)$$

In particular, the gauge symmetry of the Lagrangian L_{BF} (6.83) is

$$u = d_{\mu_1} \varepsilon_{\mu_2 \dots \mu_p} \frac{\partial}{\partial A_{\mu_1 \mu_2 \dots \mu_p}} + d_{\nu_1} \xi_{\nu_2 \dots \nu_q} \frac{\partial}{\partial B_{\nu_1 \nu_2 \dots \nu_q}}.$$

This gauge symmetry is abelian. It also is readily observed that higher-stage gauge symmetries are independent of original fields. Consequently, topological BF theory is abelian, and its gauge operator \mathbf{u} (6.92) is nilpotent. Thus, it is the BRST operator $\mathbf{b} = \mathbf{u}$. As a result, the Lagrangian L_{BF} is extended to the proper solution of the master equation $L_E = L_e$ (5.44) which reads

$$L_e = L_{\text{BF}} + \varepsilon_{\mu_2 \dots \mu_p} d_{\mu_1} \bar{A}^{\mu_1 \dots \mu_p} + \sum_{1 < k < p} \varepsilon_{\mu_{k+1} \dots \mu_p} d_{\mu_k} \bar{\varepsilon}^{\mu_k \dots \mu_p} + \varepsilon d_{\mu_p} \bar{\varepsilon}^{\mu_p} + \\ \xi_{\nu_2 \dots \nu_q} d_{\nu_1} \bar{B}^{\nu_1 \dots \nu_q} + \sum_{1 < k < q} \xi_{\nu_{k+1} \dots \nu_q} d_{\nu_k} \bar{\xi}^{\nu_k \dots \nu_q} + \xi d_{\nu_q} \bar{\xi}^{\nu_q}.$$

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